

Model Selection for Treatment Choice: Penalized Welfare Maximization

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Abstract

This paper studies a new statistical decision rule for the treatment assignment problem. Consider a utilitarian policy maker who must use sample data to allocate one of two treatments to members of a population, based on their observable characteristics. In practice, it is often the case that policy makers do not have full discretion on how these covariates can be used, for legal, ethical or political reasons. Even in cases where policy makers have leeway in how to assign treatment, plausible assumptions may generate useful constraints on treatment assignment. We treat this constrained problem as a statistical decision problem, where we evaluate the performance of decision rules by their maximum regret. We adapt and extend results from statistical learning to develop a decision rule which we call the Penalized Welfare Maximization (PWM) rule. Our study of the PWM rule, which builds on the the Empirical Welfare Maximization (EWM) rule developed in [Kitagawa and Tetenov \(2015\)](#), differs from it in two aspects. First, by imposing additional regularity conditions on the data generating process, we are able to derive bounds on the maximum regret of our rule, for a broad set of classes of treatment allocations of infinite VC dimension. In particular, we show that our rule is well suited to deal with some allocations of infinite VC dimension that can arise in applications. Second, we argue that our rule can provide a reduction in point-wise regret in situations where sample size is small compared to the complexity of the constraints on assignment. We illustrate our method in a small simulation study where our rule is able to achieve smaller regret than EWM in an empirically relevant setting. We conclude by applying our rule to data from the Job Training Partnership Act (JTPA) study.

KEYWORDS: Treatment Choice, Minimax-Regret, Statistical Learning

JEL classification codes: C01, C14, C44, C52

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1 Introduction

This paper develops a new statistical decision rule for the treatment assignment problem. A major goal of treatment evaluation is to provide policy makers with guidance on how to assign individuals to treatment, given experimental or quasi-experimental data. Following the literature inspired by [Manski \(2004\)](#) (a partial list includes [Dehejia \(2005\)](#), [Schlag \(2007\)](#), [Hirano and Porter \(2009\)](#), [Stoye \(2009\)](#), [Chamberlain \(2011\)](#), [Bhattacharya and Dupas \(2012\)](#), [Tetenov \(2012\)](#), [Stoye \(2012\)](#), [Kasy et al. \(2014\)](#), [Kitagawa and Tetenov \(2015\)](#) provide an excellent review), we treat the treatment assignment problem as a statistical decision problem of maximizing population welfare. Like many of the above papers, we evaluate our decision rule by its maximum regret.

Often, policy makers have observable characteristics at their disposal on which to base treatment, however, they may not always have full discretion on how these covariates can be used. For example, policy makers may face exogenous constraints on how they can use covariates for legal, ethical, or political reasons. Even in cases where policy makers have leeway in how they assign treatment, plausible assumptions may imply certain restrictions on assignment. The rule we propose, which we call the Penalized Welfare Maximization (or PWM) rule, is designed for these situations. An essential building block and motivation for our rule is the Empirical Welfare Maximization (or EWM) rule of [Kitagawa and Tetenov \(2015\)](#). A primary feature of EWM is its ability to solve the treatment choice problem when exogenous constraints are placed on treatment assignment. In the machine learning literature, [Beygelzimer and Langford \(2009\)](#), [Kallus \(2016\)](#), and [Zadrozny \(2003\)](#) adapt popular techniques such as support vector machines and trees/forests to solve the unrestricted version of this problem.

Our rule is designed to address two specific situations related to treatment choice with exogenous constraints. The first situation we address is when the constraints placed on assignment do not satisfy the conditions in [Kitagawa and Tetenov \(2015\)](#). To be concrete, suppose we have two treatments, and we denote assignment into these treatments by partitioning the covariate space into two pieces. We can then think of constraints on assignment as constraints on the allowable subsets we can consider for the partition. [Kitagawa and Tetenov \(2015\)](#) derive bounds on maximum regret for the EWM rule when the class of allowable subsets has finite VC dimension (see [Györfi et al. \(1996\)](#) for a definition). Although constraints of this form can be quite flexible, we will present examples where the constraints we consider do not have finite VC dimension. In particular, we will argue that the constraints imposed by some reasonable assumptions may generate classes of infinite VC dimension. Bounds on the maximum regret of EWM do exist for *some* classes of infinite VC dimension under additional regularity conditions: for example [Tsybakov \(2004\)](#) derives bounds on the classification-analogue of EWM when the class has a sufficiently small bracketing entropy (see

⁰(continued from previous page) for helpful comments, as well as Nitish Keskar for help in implementing EWM. This research was supported in part through the computational resources and staff contributions provided for the Social Sciences Computing Cluster (SSCC) at Northwestern University. All mistakes are our own.

also Proposition 5.2), but there exist examples of classes that do not satisfy these conditions. To solve this problem, we approximate classes of infinite VC dimension by sequences of classes of finite VC dimension in which EWM can be applied. The strength of the PWM rule in this application will then be to provide a data-driven method by which to select the appropriate approximating class in applications. In doing so we will derive bounds on the maximum regret of the PWM rule for classes of infinite VC dimension. We go on to apply this result to data from the Job Training Partnership Act (JTPA) study.

The second situation we consider is when the constraints placed on assignment *do* satisfy the conditions in Kitagawa and Tetenov (2015), but the size of the sample on which to estimate the rule is relatively small. As is shown in their paper, when the constraints placed on assignment are too flexible relative to the sample size available, the EWM rule may suffer from overfitting, which can result in inflated values of regret. By the same mechanism that allows PWM to select an appropriate approximating class in our first application, we can use PWM in an attempt to reduce *point-wise* regret in situations where the resulting regret of EWM is large. We illustrate PWM's ability to reduce regret in a simulation study where the policy maker has many covariates on which to assign treatment, but does not know how many to use in practice.

The PWM rule is heavily inspired by the literature on model selection in classification (see for example Györfi et al. (1996), Koltchinskii (2001), Bartlett et al. (2002), Scott and Nowak (2002), Boucheron et al. (2005), Bartlett (2008), Koltchinskii (2008)). The theoretical contribution of our paper is to modify and extend some of these tools to the setting of treatment choice. As pointed out in Kitagawa and Tetenov (2015), there are substantive differences between classification and treatment choice: observed outcomes are real-valued in the setting of treatment choice, and only one of the potential outcomes is observed for any given individual. When we say that we extend these tools, we mean that we prove results for settings where the data available to the policy maker is observational or quasi-experimental. As we will see, in such a setting the policy maker's objective function contains an estimated quantity, which is not an issue that arises in the classification problem. In deciding which tools to extend, we have attempted to strike a balance between ease of use for practitioners, theoretical appeal, and performance in simulations.

The remainder of the paper is organized as follows. In Section 2, we setup the notation and formally define the problem that the policy maker (i.e. social planner) is attempting to solve. In Section 3, we introduce the PWM rule and present general results about its maximum (and pointwise) regret. In Section 4, we perform a small simulation study to highlight the ability for PWM to reduce regret in settings where sample size is relatively small. In Section 5 we derive bounds on maximum regret of the PWM rule when the planner is constrained to what we call *monotone* allocations, and then illustrate these in an application to the JTPA study. Section 6 concludes.

2 Setup and Notation

Let Y_i denote the observed outcome of a unit i , and let D_i be a binary variable which denotes the treatment received by unit i . Let $Y_i(1)$ denote the *potential* outcome of unit i under treatment 1 (which we will sometimes refer to as “the treatment”), and let $Y_i(0)$ denote the potential outcome of unit i under treatment 0 (which we will sometimes refer to as “the control”). The observed outcome for each unit is related to their potential outcomes through the identity:

$$Y_i = Y_i(1)D_i + Y_i(0)(1 - D_i) . \quad (1)$$

Let $X_i \in \mathcal{X} \subset \mathbb{R}^{d_x}$ denote a vector of observed covariates for unit i . Let Q denote the distribution of $(Y_i(0), Y_i(1), D_i, X_i)$, then we assume that the planner observes a size n random sample

$$(Y_i, D_i, X_i)_{i=1}^n \sim P^n ,$$

where P is jointly determined by Q , and the identity in (1). Throughout the paper we will assume unconfoundedness, i.e.

Assumption 2.1. (*Unconfoundedness*) *The distribution Q satisfies:*

$$\left((Y(1), Y(0)) \perp D \right) \mid X .$$

This assumption asserts that, once we condition on the observable covariates, the treatment is exogenous. This assumption will hold in a randomized controlled trial (RCT), which is our primary application of interest, since the treatment is exogenous by construction. This assumption is sometimes also made (possibly tenuously) in observational studies; it is a key identifying assumption when using matching or regression estimators in policy evaluation settings with observational data (Imbens (2004) provides a review of these techniques, and discusses the validity of Assumption 2.1 in economic applications).

The planner’s goal is to optimally assign the treatment to the population. The objective function we consider is utilitarian welfare, which is defined by the average of the individual outcomes in the population:

$$E_Q[Y(1)\mathbf{1}\{X \in G\} + Y(0)\mathbf{1}\{X \notin G\}] ,$$

where $G \subset \mathcal{X}$ represents the covariate values for those individuals to whom treatment 1 is assigned. The planner is tasked with choosing a *treatment allocation* $G \subset \mathcal{X}$ using the empirical data. Using Assumption 2.1, we can rewrite the welfare criterion as:

$$E_Q[Y(0)] + E_P\left[\left(\frac{YD}{e(X)} - \frac{Y(1-D)}{1-e(X)}\right)\mathbf{1}\{X \in G\}\right] ,$$

where $e(X) = E_P[D|X]$ is the propensity score. Since the first term of this expression does not depend on G , we define the planner’s objective function given a choice of treatment allocation G

as:

$$W(G) := E_P \left[\left(\frac{YD}{e(X)} - \frac{Y(1-D)}{1-e(X)} \right) \mathbf{1}\{X \in G\} \right].$$

Let \mathcal{G} be the class of feasible treatment allocations $G \in \mathcal{G}$. Here when we say feasible, we mean that it may be the case that the planner is restricted in what kinds of allocations they can (or want to) consider. For instance, it could be the case that the planner is not able to select certain treatment allocations for legal, ethical, or political reasons, or it could be that a specific application justifies certain types of allocations. Consider the following three examples of \mathcal{G} :

Example 2.1. \mathcal{G} could be the set of all measurable subsets of \mathcal{X} . This is the largest possible class of admissible allocations. It is straightforward to show that the optimal allocation in this case is as follows: define

$$\tau(x) := E_Q[Y(1) - Y(0)|X = x],$$

then the optimal allocation is given by

$$G_{FB}^* := \{x \in \mathcal{X} : \tau(x) \geq 0\}.$$

■

Example 2.2. Suppose $\mathcal{X} \subset \mathbb{R}$, and consider the class of *threshold allocations*:

$$\mathcal{G} = \{G : G = (-\infty, x] \cap \mathcal{X} \text{ or } G = [x, \infty) \cap \mathcal{X}, \text{ for } x \in \mathcal{X}\}.$$

Such a class \mathcal{G} could be reasonable, for example, when assigning scholarships to students: suppose the only covariate available to the planner is a student's GPA, then it may be school policy that only threshold-type rules are to be considered. ■

Example 2.3. Suppose $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \subset \mathbb{R}^2$, and consider the class of *monotone allocations*:

$$\mathcal{G} = \{G : G = \{(x_1, x_2) \in \mathcal{X} \mid x_2 \geq f(x_1) \text{ for } f : \mathcal{X}_1 \rightarrow \mathcal{X}_2 \text{ increasing}\}\}.$$

As an example, consider again the setting of assigning scholarships to students, but now suppose that the covariates available to the planner are a student's GPA and parental income. It could then be school policy that the treatment allocation be such that the GPA requirement for a scholarship increases the higher the student's parental income. In fact, even if the planner were *not* exogenously constrained to such an allocation, it may be the case that reasonable assumptions *justify* the use of such a restriction. For example, suppose that the outcome of interest depends only on a student's innate "ability", which is unobservable, and whether or not they receive a scholarship. Further suppose that the planner must use information on GPA and parental income to assign scholarships, which have a per-unit cost. Under some assumptions about the outcome equation, and the relationship between the distributions of ability, GPA, and parental income, it can be shown that the optimal allocation is in fact monotone. In Appendix B we work through this example in detail. ■

Given an admissible class \mathcal{G} , we denote the highest attainable welfare by:

$$W_{\mathcal{G}}^* := \sup_{G \in \mathcal{G}} W(G) .$$

A *decision rule* is a function \hat{G} from the observed data $\{(Y_i, D_i, X_i)\}_{i=1}^n$ into the set of admissible allocations \mathcal{G} . We call the rule that we develop and study in this paper the *Penalized Welfare Maximization* (or PWM) rule. As in much of the literature that follows the work of [Manski \(2004\)](#), we assume that the planner is interested in rules \hat{G} that, on average, are close to the highest attainable welfare. To that end, the criterion by which we evaluate a decision rule is given by what we call maximum \mathcal{G} -regret:

$$\sup_P E_{P^n} [W_{\mathcal{G}}^* - W(\hat{G})] .$$

It is important to note that most of the papers on statistical treatment rules that are concerned with maximum regret consider only regret relative to the unrestricted optimum. [Kitagawa and Tetenov \(2015\)](#) were the first to consider regret relative to the second-best optimum, as we do here. As they discuss, this restriction is important for their results because it allows them to establish a rate of convergence for the maximum regret of their decision rule. We will see in [Section 3](#) that such a restriction is important for the results that we derive as well. However, we will see in [Example 3.3](#) that, by imposing suitable regularity conditions, our rule could also be useful when the planner is unrestricted.

3 Penalized Welfare Maximization

In this section, we present the main results of our paper. In [Section 3.1](#), we review the properties of the empirical welfare maximization (EWM) rule of [Kitagawa and Tetenov \(2015\)](#), which will motivate the PWM rule and serve as an important building block in its construction. In [Section 3.2](#), we define the penalized welfare maximization rule and present bounds on its maximum \mathcal{G} -regret for general penalties. In [Section 3.3](#) we illustrate these results by applying them to some specific penalties. Finally, in [Section 3.4](#) we present results for a modification of the PWM rule for applications where the propensity score is not known and must be estimated.

3.1 Empirical Welfare Maximization: a Review and Some Motivation

The idea behind the EWM rule is to solve a sample analog of the population welfare maximization problem:

$$\hat{G}_{EWM} \in \arg \max_{G \in \mathcal{G}} W_n(G) ,$$

where

$$W_n(G) := \frac{1}{n} \sum_{i=1}^n \tau_i \mathbf{1}\{X_i \in G\} := \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{Y_i D_i}{e(X_i)} - \frac{Y_i (1 - D_i)}{1 - e(X_i)} \right) \mathbf{1}\{X_i \in G\} \right] . \quad (2)$$

Note that to solve this optimization problem, the planner must know the propensity score $e(\cdot)$. This assumption is reasonable if the data comes from a randomized experiment, but clearly could not be made in a setting where the planner is using observational data. [Kitagawa and Tetenov \(2015\)](#) derive results for a modified version of the EWM rule where the propensity score is estimated, which we will review in Section [3.4](#).

To derive their non-asymptotic bounds on the maximum \mathcal{G} -regret of the EWM rule, [Kitagawa and Tetenov \(2015\)](#) make the following additional assumptions, which we will also require for our results:

Assumption 3.1. (*Bounded Outcomes and Strict Overlap*) *The set of distributions $\mathcal{P}(M, \kappa)$ has the following properties:*

- *There exists some $M < \infty$ such that the support of the outcome variable Y is contained in $[-\frac{M}{2}, \frac{M}{2}]$.*
- *There exists some $\kappa \in (0, 0.5)$ such that $e(x) \in [\kappa, 1 - \kappa]$ for all x .*

The first assumption asserts that the outcome is bounded. Since the implementation of both the EWM rule and PWM rule do not require that the planner knows M , and the existence of *some* bound on outcomes of interest to economics seems plausible, we feel that this assumption is tenable. The second assumption is standard when imposing unconfoundedness. In a RCT, this assumption will hold by design, but may be violated in settings with observational data.

In order to derive their results, [Kitagawa and Tetenov \(2015\)](#) also make the following assumption, which we will *not* require:

Assumption 3.2. (*Finite VC Dimension*)¹ : \mathcal{G} has finite VC dimension $V < \infty$.

Such an assumption may or may not be restrictive depending on the application in question. Consider Example [2.2](#), the class of threshold allocations on \mathbb{R} . This class has VC dimension 2, and so Assumption [3.2](#) holds. On the other hand, it can be shown that the class of monotone allocations on $[0, 1]^2$ that was introduced in Example [2.3](#) has infinite VC dimension (see [Györfi et al. \(1996\)](#)), so that the bounds derived in [Kitagawa and Tetenov \(2015\)](#) will not apply.

Given Assumptions [3.1](#) and [3.2](#), [Kitagawa and Tetenov \(2015\)](#) derive the following non-asymptotic upper bound on the maximum \mathcal{G} -regret of the EWM rule:

$$\sup_{P \in \mathcal{P}(M, \kappa)} E_{P^n} [W_{\mathcal{G}}^* - W(\hat{G}_{EWM})] \leq C \frac{M}{\kappa} \sqrt{\frac{V}{n}}, \quad (3)$$

¹It should be possible to derive analogous results by assuming that the class of treatment allocations has sufficiently small bracketing entropy (as in [Tsybakov \(2004\)](#)). We will also not require such an assumption.

for some universal constant C . Moreover, when X has sufficiently large support, they derive the following *lower* bound for *any* decision rule \hat{G} :

$$\sup_{P \in \mathcal{P}(M, \kappa)} E_{P^n}[W_{\mathcal{G}}^* - W(\hat{G})] \geq RM \sqrt{\frac{V-1}{n}}, \quad (4)$$

for R a universal constant and n sufficiently large. This shows that the rate of convergence of maximum \mathcal{G} -regret implied by (3) is the best possible, i.e. that no other decision rule could achieve a faster rate without imposing additional assumptions.

Remark 3.1. In fact, Theorem 2.2 in Kitagawa and Tetenov (2015), which establishes (4), implies another interesting result: if X is continuously distributed and we do not impose additional restrictions on the distribution P , then it is *impossible* to derive a uniform rate on maximum \mathcal{G} -regret for *any* rule, for classes \mathcal{G} of infinite VC dimension. This is in line with the results derived in Stoye (2009), where he shows that, for any sample size, flipping a coin to assign individuals is minimax-regret optimal despite this rule not even being pointwise consistent. Since we will be interested in classes \mathcal{G} such as the one presented in Example 2.3, we will revisit this problem later in Section 3. ■

Remark 3.2. As pointed out in Kitagawa and Tetenov (2015), the EWM rule is not invariant to positive affine transformations of the outcomes, and thus the researcher could manipulate the treatment rule in settings where they have leeway in how to code the outcome variable. To deal with this issue they suggest solving a demeaned version of the welfare maximization problem. In Appendix B we discuss the demeaned version of EWM and redo the exercises of Sections 4 and 5 using a demeaned version of EWM and PWM. ■

We are now ready to pose the two problems we focus on in this paper for which the PWM rule is designed. The first problem we pose is that the bound in (3) does not apply to situations in which \mathcal{G} has infinite VC dimension. Although this may not be problematic in some applications, Example 2.3 highlights a situation where classes of infinite VC dimension could be relevant in practice. We will consider situations where it is possible to “approximate” \mathcal{G} with a sequence of classes of finite VC dimension,

$$\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{G}_3 \subset \dots \subset \mathcal{G}_k \subset \dots$$

Let $\hat{G}_{n,k}$ be the EWM rule in the class \mathcal{G}_k . Then we can decompose the \mathcal{G} -regret of the rule $\hat{G}_{n,k}$ as follows:

$$E_{P^n}[W_{\mathcal{G}}^* - W(\hat{G}_{n,k})] = E_{P^n}[W_{\mathcal{G}_k}^* - W(\hat{G}_{n,k})] + W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^*.$$

Given this decomposition, we call

$$E_{P^n}[W_{\mathcal{G}_k}^* - W(\hat{G}_{n,k})],$$

the *estimation* error of the rule $\hat{G}_{n,k}$ in the class \mathcal{G}_k , and we call

$$W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^*,$$

the *approximation* error of the class \mathcal{G}_k . Typically, we would expect that the higher is k , the larger the estimation error, and the smaller the approximation error. Suppose it was the case that we could derive sharp uniform bounds on these errors, then an appropriate choice of k would be one that balances the tradeoff according these bounds. As we will see in Corollary 3.1, the power of the PWM rule will be that it will choose k appropriately, and in a data-driven fashion. In Section 5, we apply the PWM rule to an application where the class we consider could have infinite VC dimension.

The second problem we address is when the sample size is small, but the class \mathcal{G} is relatively complex. The bound on regret given by (3) is worse the larger is V and the smaller is n : this is because of the ability for complex classes \mathcal{G} to “overfit” the data in small samples. In a situation where V is large relative to n , it may be beneficial to perform EWM in a class \mathcal{G}' of smaller VC dimension, so that the bound on

$$E_{P^n}[W_{\mathcal{G}'}^* - W(\hat{G}_{EWM})] ,$$

is small. On the other hand, this will only be useful if it is also the case that

$$W_{\mathcal{G}}^* - W_{\mathcal{G}'}^* ,$$

is small as well: here we face the same tradeoff between estimation and approximation error as we did above. As we will see in Theorem 3.1 and Corollary 3.2, PWM will be able to find the right class in this setting as well. It is important to note that the benefit of PWM over EWM for this type of application comes through a potential reduction in *pointwise* \mathcal{G} -regret: as we have seen from (4), there is no sense in which we can improve on the EWM rule in a uniform way in such an application. In Section 4, we apply the PWM rule in a simulation study where we are able to achieve a reduction in pointwise regret relative to the EWM rule.

3.2 Penalized Welfare Maximization: General Results

We are now ready to introduce the PWM rule. Unlike in Section 3.1 where we considered an admissible class \mathcal{G} of finite VC dimension, in this section we do not specify the VC dimension of \mathcal{G} in general. Instead, we consider the following nested sequence of classes which will serve as a *sieve sequence* of \mathcal{G} :

Assumption 3.3. *The sequence of classes*

$$\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \dots, \mathcal{G}_k, \dots$$

*is such that each class $\mathcal{G}_k \subseteq \mathcal{G}$ has VC dimension V_k , which is finite.*²

Interesting sequences $\{\mathcal{G}_k\}_k$ may be finite or countable. We illustrate this with some examples:

Example 3.1. Recall the class of threshold allocations introduced in Example 2.2. Let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \subset \mathbb{R}^2$, and define \mathcal{G}_X^1 to be the threshold allocations on \mathcal{X}_1 and \mathcal{G}_X^2 to be the threshold allocations on \mathcal{X}_2 . We can now define the set of *two-dimensional* threshold allocations on \mathcal{X} :

$$\mathcal{G} = \{G \subset \mathcal{X} : G = G_1 \times G_2, \ G_1 \in \mathcal{G}_X^1 \text{ and } G_2 \in \mathcal{G}_X^2\}.$$

To make this concrete, suppose \mathcal{X}_1 is an age covariate and \mathcal{X}_2 is an income covariate, then this class contains allocations of the form, for example, “receive treatment if age is above x_1 and income is below x_2 ” for some x_1 and x_2 .

In practice, it may not be obvious which (if any) of these covariates should be used to assign treatment: it may be the case that one of these covariates is not useful for treatment allocation, or that the sample size is small relative to the VC dimension of \mathcal{G} . In these situations, an interesting sieve sequence for \mathcal{G} is given by the following: let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be defined as

$$\mathcal{G}_1 = \{\emptyset, \mathcal{X}\}, \ \mathcal{G}_2 = (\mathcal{G}_X^1 \otimes \mathcal{X}_2) \cup (\mathcal{X}_1 \otimes \mathcal{G}_X^2), \ \mathcal{G}_3 = \mathcal{G},$$

where

$$\mathcal{G}_X^1 \otimes \mathcal{X}_2 := \{G \times \mathcal{X}_2 : G \in \mathcal{G}_X^1\}, \ \mathcal{X}_1 \otimes \mathcal{G}_X^2 := \{\mathcal{X}_1 \times G : G \in \mathcal{G}_X^2\}.$$

The sequence $\{\mathcal{G}_k\}_{k=1}^3$ corresponds to the sequence of threshold allocations that use zero, one and two covariates respectively (that each class \mathcal{G}_k has finite VC dimension follows from the fact a class of threshold allocations in one dimension has finite VC dimension, and that unions of classes of finite VC dimension have finite VC dimension, see [Dudley \(1999\)](#)).³ We will revisit this example in the simulation study of Section 4, where we will consider threshold allocations in 5 dimensions along with the corresponding sequence of classes that cut on zero, one, two, three, four and five dimensions. ■

Example 3.2. Recall the class of monotone allocations introduced in Example 2.3. Suppose that $\mathcal{X} = [0, 1]^2$, so that \mathcal{G} has infinite VC dimension (see [Györfi et al. \(1996\)](#) for a proof of this fact). We will construct a useful sieve sequence for \mathcal{G} , where we approximate sets in \mathcal{G} with sets that feature monotone, piecewise-linear boundaries. We proceed in three steps.

First define, for T an integer and $0 \leq j \leq T$, the following function $\psi_{T,j} : [0, 1] \rightarrow [0, 1]$:

$$\psi_{T,j}(x) = \begin{cases} 1 - |Tx - j|, & x \in [\frac{j-1}{T}, \frac{j+1}{T}] \cap [0, 1] \\ 0, & \text{otherwise} . \end{cases}$$

²[Kitagawa and Tetenov \(2015\)](#) additionally assume that their class \mathcal{G} is countable so as to avoid potential measurability concerns. We instead choose not to address these concerns explicitly, as is done in most of the literature on classification. See [Van Der Vaart and Wellner \(1996\)](#) for a discussion of possible resolutions to this issue.

³Note that in this example, it is actually the case that \mathcal{G}_2 and \mathcal{G}_3 have the same VC dimension. This will not be the case when we move to the setting in 5 dimensions.

The function $\psi_{T,j}(\cdot)$ is simply a triangular kernel whose base shifts with j and is scaled by T . For example, $\psi_{4,1}(\cdot)$ is a triangular kernel with base $[0, 0.5]$, and $\psi_{8,1}(\cdot)$ is a triangular kernel with base $[0, 0.25]$. Next, using these functions, we define the following classes \mathcal{S}_k :

$$\mathcal{S}_k = \left\{ G : G = \{x = (x_1, x_2) \in \mathcal{X} \mid \sum_{j=0}^T \theta_j \psi_{T,j}(x_1) + x_2 \geq 0\} \text{ for } \theta_j \in \mathbb{R}, \forall 0 \leq j \leq T \right\},$$

where $T = 2^{k-1}$. These \mathcal{S}_k are a special case of what Kitagawa and Tetenov (2015) call *generalized eligibility scores*, which, as shown in Dudley (1999), have VC dimension $T+2$. The intuition behind the class \mathcal{S}_k is that it divides the covariate space into treatment and control such that the boundary is a piecewise linear curve. Note that by construction it is the case that $\mathcal{S}_{k-1} \subset \mathcal{S}_k$ for every k . Finally, to construct our approximating class \mathcal{G}_k , we will modify the class \mathcal{S}_k such that we ensure that the resulting treatment allocations are monotone.

For T an integer, let D_T be the following $T \times (T+1)$ *differentiation matrix*:

$$D_T := \begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}.$$

Then \mathcal{G}_k is defined as follows:

$$\mathcal{G}_k = \left\{ G : G \in \mathcal{S}_k \text{ and } D_T \Theta_T \geq 0, \Theta_T = [\theta_0 \dots \theta_T]' \right\},$$

for $T = 2^{k-1}$. Note that the purpose of the constraint $D_T \Theta_T \geq 0$ is to ensure that $\theta_k - \theta_{k-1} \geq 0$ for all k , which is what imposes monotonicity on the allocations. This construction, introduced by Beresteanu (2004) in the context of regression, is useful as it imposes monotonicity through a *linear* constraint, which is ideal for our implementation of this sequence in Section 5. Proposition 5.1 provides a uniform rate at which $W_{\mathcal{G}_k}^* \rightarrow W_{\mathcal{G}}^*$ under some additional regularity conditions, and Corollary 5.1 derives the corresponding bound on maximum \mathcal{G} -regret of the PWM rule. It is important to mention that, under the regularity conditions we will impose, the class of monotone allocations is an example of a class for which bounds on maximum \mathcal{G} -regret exist for EWM, despite this class having infinite VC dimension (see Proposition 5.2). We will compare the bounds we derive for PWM to these bounds in the discussion following Corollary 5.1. In Section 5, we study the use of this sequence of approximating classes in an application to the JTPA study. ■

Example 3.3. Suppose the planner faces no restrictions on treatment assignment, so that \mathcal{G} is the class of all measurable subsets of \mathcal{X} . Recall from Example 2.1 that the optimal allocation in this case is given by G_{FB}^* . In this setting it may seem natural to employ the *plug-in* decision rule:

$$\hat{G}_{plug-in} := \{x : \hat{\tau}(x) \geq 0\},$$

where $\hat{\tau}(\cdot)$ is a non-parametric estimate of $\tau(\cdot)$. Under Assumption 2.1 many non-parametric estimates of $\tau(\cdot)$ are well understood (see Imbens (2004) for a review). The Penalized Welfare Maximization Rule could provide an interesting alternative to the plug-in rule in this setting by considering a sequence of sieve classes that form a *decision tree*. Decision trees are popular rules in classification because of their natural interpretability. Intuitively, a decision tree recursively partitions the covariate space in such a way that the resulting decision rule can be understood as a series of “yes-or-no” questions involving the covariates. Using decision trees for the estimation of causal effects has recently become a popular idea (see for example Athey and Imbens (2015), Wager and Athey (2015)). Although we do not explore decision trees further in this paper, in Appendix B we explain how we could accommodate them in our framework and relate them to the development of Optimal Personalization Trees in Kallus (2016). We also provide a preliminary comparison to a plug-in decision rule. ■

Recall the two motivations we presented for the PWM rule at the end of Section 3.1. Given a sieve sequence $\{\mathcal{G}_k\}_k$, let

$$\hat{G}_{n,k} := \arg \max_{G \in \mathcal{G}_k} W_n(G) ,$$

be the EWM rule in the class \mathcal{G}_k . Our goal is to select the appropriate class k^* in which to perform EWM, either because we are approximating a large class \mathcal{G} , or because we would like to reduce \mathcal{G} -regret in small samples. To that end, recall our decomposition of \mathcal{G} -regret of the rule $\hat{G}_{n,k}$:

$$E_{P^n}[W_{\mathcal{G}}^* - W(\hat{G}_{n,k})] = E_{P^n}[W_{\mathcal{G}_k}^* - W(\hat{G}_{n,k})] + W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^* ,$$

The idea of the PWM rule is to select some k^* such that the \mathcal{G} -regret of the resulting rule achieves the optimal balance between

$$E_{P^n}[W_{\mathcal{G}_{k^*}}^* - W(\hat{G}_{n,k^*})] ,$$

which we called the estimation error, and

$$W_{\mathcal{G}}^* - W_{\mathcal{G}_{k^*}}^* ,$$

which we called the approximation error.

We do this by selecting the class k^* in the following way: for each class \mathcal{G}_k , suppose we had some measure $C_n(k)$ of the amount of “overfitting” that results from using the rule $\hat{G}_{n,k}$ (we will be more precise about the nature of $C_n(k)$ in a moment). Given such a measure $C_n(k)$, let $\{t_k\}_{k=1}^\infty$ be any increasing sequence of real numbers, and define the following penalized objective function:

$$R_{n,k}(G) := W_n(G) - C_n(k) - \sqrt{\frac{t_k}{n}} .$$

Then the *penalized welfare maximization* rule \hat{G}_n is the solution to:

$$\hat{G}_n := \arg \max_k R_{n,k}(\hat{G}_{n,k}) .$$

Remark 3.3. Note that the PWM objective function $R_{n,k}(\cdot)$ includes a $\sqrt{t_k/n}$ term. This component of the objective is required for technical reasons when the approximating sequence $\{\mathcal{G}_k\}_k$ is infinite, as it ensures that the classes get penalized at a sufficiently fast rate as k increases. Ideally, we would like the penalty to be completely intrinsic to each class, but this technical device seems to be unavoidable and similar devices are pervasive throughout the literature on model selection in classification: see [Koltchinskii \(2001\)](#), [Bartlett et al. \(2002\)](#), [Boucheron et al. \(2005\)](#), [Koltchinskii \(2008\)](#). We will make three comments about this term. First of all, in our experience, this added term is so small relative to the penalty $C_n(k)$ that its presence is irrelevant when performing PWM in applications. Second, if one is only interested in using PWM to reduce pointwise regret in settings where the sequence of classes is finite, then the $\sqrt{t_k/n}$ term is not required. Third, our results hold for *any* increasing sequence $\{t_k\}_{k=1}^\infty$. For clarity, unless otherwise specified, we will present all of our results with this term included, and set $t_k = k$. ■

Remark 3.4. It is worth commenting on why such a penalty is necessary in the first place. It seems reasonable that, given a sieve sequence $\{\mathcal{G}_k\}_k$, one could construct a sequence $k(n)$ such that the decision rule $\hat{G}_{n,k(n)}$ achieves an optimal balance between the estimation and approximation error. However, it is impossible to construct such a sequence that would apply to every set of admissible rules \mathcal{G} , approximating sequence $\{\mathcal{G}_k\}_k$, and class of data generating processes; the appropriate $k(n)$ would depend on the VC dimensions V_k of the sieve sequence, which may be hard to bound, as well as knowledge of the uniform rate of convergence of $W_{\mathcal{G}_k}^*$ to $W_{\mathcal{G}}^*$, which will depend on \mathcal{G} , $\{\mathcal{G}_k\}_k$, and the regularity conditions we are willing to impose on P . Our method will provide a way to find the appropriate class in which to maximize in general. Moreover, recall that we are also interested in trying to reduce pointwise \mathcal{G} -regret in situations where the VC dimension of \mathcal{G} is large relative to sample size. We will see in [Theorem 3.1](#) and [Corollary 3.2](#) that our penalization method, being data-driven, achieves this as well. ■

Before stating our main results about the \mathcal{G} -regret of the PWM rule, we must formalize how $C_n(k)$ should behave. In this section we will present high level assumptions that the penalty $C_n(k)$ must possess, and in [Section 3.3](#) we will provide specific examples. We make the following assumption on the penalty $C_n(k)$:

Assumption 3.4. *There exist positive constants c_0 and c_1 such that $C_n(k)$ satisfies the following tail inequality for every n, k , and for every $\epsilon > 0$.⁴*

$$\sup_{P \in \mathcal{P}(M, \kappa)} P^n(W_n(\hat{G}_{n,k}) - W(\hat{G}_{n,k}) - C_n(k) > \epsilon) \leq c_1 e^{-2c_0 n \epsilon^2}.$$

We will provide some intuition for this assumption. Given an EWM rule $\hat{G}_{n,k}$, the value of the empirical welfare is given by $W_n(\hat{G}_{n,k})$. From the perspective of \mathcal{G} -regret, what we would really like

⁴If one were only interested in point-wise regret bounds, this assumption could be weakened to allow for penalties for which the inequalities hold point-wise in P , instead of uniformly.

to know is the value of population welfare $W(\hat{G}_{n,k})$. Although this is not knowable, suppose we could define $C_n(k)$ as $W_n(\hat{G}_{n,k}) - W(\hat{G}_{n,k})$, then the penalized objective $W_n(\hat{G}_{n,k}) - C_n(k)$ would be exactly $W(\hat{G}_{n,k})$. Since implementing such a $C_n(k)$ is impossible, we require our penalty to be a good upper bound on this quantity to obtain our results. We are now ready to state our main workhorse result about the \mathcal{G} -regret of the PWM rule:

Theorem 3.1. *Consider Assumptions 2.1, 3.1, 3.3 and 3.4, and fix $\{t_k\}_{k=1}^\infty$ such that $t_k = k$. Then there exist constants Δ and c_0 such that for every $P \in \mathcal{P}(M, \kappa)$:*

$$E_{P^n}[W_{\mathcal{G}}^* - W(\hat{G}_n)] \leq \inf_k \left[E_{P^n}[C_n(k)] + (W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^*) + \sqrt{\frac{k}{n}} \right] + \sqrt{\frac{\log(\Delta e)}{2c_0 n}}.$$

This result forms the basis of all the results we present in Sections 3.2 and 3.3. It says that, at least from the perspective of *pointwise* \mathcal{G} -regret, the PWM rule is able to balance the tradeoff between $E_{P^n}[C_n(k)]$ and the approximation error, at the cost of adding two additional terms that are $O(1/\sqrt{n})$. We comment on the nature of these terms in Remark 3.5 below. Note that this result as stated does not *quite* accomplish our initial goal of balancing the estimation and approximation error along our sieve sequence: it is possible to choose a $C_n(k)$ that satisfies Assumption 3.4 for which $E_{P^n}[C_n(k)]$ is too large. For this reason, we also impose the requirement that any penalty we consider should have the following additional property:

Assumption 3.5. *There exists a positive constant C_1 such that, for every n , $C_n(k)$ satisfies*

$$\sup_{P \in \mathcal{P}(M, \kappa)} E_{P^n}[C_n(k)] \leq C_1 \sqrt{\frac{V_k}{n}},$$

where V_k is the VC dimension of \mathcal{G}_k .

This assumption ensures that $E_{P^n}[C_n(k)]$ is comparable to the estimation error for EWM derived in (3).

Remark 3.5. Theorem 3.1 shows that the PWM rule is able to balance the tradeoff between the estimation and approximation errors, but the bound we derive introduces two additional terms. The second of these terms, at this level of generality, is hard to quantify. We will attempt to shed light on this term, for specific penalties, in Section 3.3. ■

The next result we present is meant to address the first problem we posed at the end of Section 3.1. It shows that, if there exists a uniform bound on the approximation error, then the maximum regret of the PWM rule is such that we select the class k^* appropriately. First we make the assumption that we restrict ourselves to a set of distributions \mathcal{P}_r for which there exists a uniform bound on the approximation error:

Assumption 3.6. Let \mathcal{P}_r be a set of distributions such that

$$\sup_{P \in \mathcal{P}_r} W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^* = O(\gamma_k) ,$$

$$\sup_{P \in \mathcal{P}_r \cap \mathcal{P}(M, \kappa)} E_{P^n} [C_n(k)] = O(\zeta(k, n)) ,$$

for a sequence $\gamma_k \rightarrow 0$, and $\zeta(k, n)$ non-decreasing as $k \rightarrow \infty$, $\zeta(k, n) \rightarrow 0$ as $n \rightarrow \infty$.

Note that the first assumption asserts that we have a uniform bound on the approximation error. As we pointed out in Remark 3.1, an assumption of this type is necessary to derive a bound on maximum regret when the class \mathcal{G} has infinite VC dimension. The second assumption is made to highlight the following possibility: although Assumption 3.5 guarantees that we can satisfy this restriction with $\zeta(k, n) = \sqrt{V_k/n}$, it is possible that, once we have imposed that P must lie in \mathcal{P}_r , an even *tighter* bound may exist on $C_n(k)$. We make this point to emphasize that PWM will balance the tradeoff between the estimation and approximation error according to the *tightest* possible bounds on $E_P[C_n(K)]$ and $W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^*$, regardless of whether or not we know these bounds.

Corollary 3.1. Under Assumptions 2.1, 3.1, 3.3, 3.4, and 3.6, we have that

$$\sup_{P \in \mathcal{P}_r \cap \mathcal{P}(M, \kappa)} E_{P^n} [W_{\mathcal{G}}^* - W(\hat{G}_n)] \leq \inf_k \left[O(\zeta(k, n)) + O(\gamma_k) + \sqrt{\frac{k}{n}} \right] + \sqrt{\frac{\log(\Delta e)}{2c_0 n}} .$$

As mentioned in Remark 3.4, if $\{\zeta(k, n)\}_{k,n}$ and $\{\gamma_k\}_k$ were known, then we could achieve such a result with a deterministic sequence $k(n)$. The strength of the PWM rule then is that we achieve the *same* behavior for any class \mathcal{G} and approximating sequence $\{\mathcal{G}_k\}_k$ without having to know these quantities in practice. We will illustrate this result in our application Section 5, in the setting of Example 3.2.

Our final result of Section 3.2 is meant to address the second problem we posed at the end of Section 3.1. It restates Theorem 3.1 in such a way that it highlights the ability of the PWM rule to reduce *pointwise* regret:

Corollary 3.2. Consider Assumptions 2.1, 3.1, 3.3, 3.4, and 3.5. Suppose

$$\mathcal{G} = \bigcup_k \mathcal{G}_k ,$$

and suppose that P is such that $W_{\mathcal{G}}^*$ is achieved by some $G^* \in \mathcal{G}$. Let k^* be the smallest k such that

$$G^* \in \mathcal{G}_{k^*} ,$$

then we have that

$$E_{P^n} [W_{\mathcal{G}}^* - W(\hat{G}_n)] \leq C_1 \sqrt{\frac{V_{k^*}}{n}} + \sqrt{\frac{k^*}{n}} + \sqrt{\frac{\log(\Delta e)}{2c_0 n}} .$$

Furthermore, if $\{\mathcal{G}_k\}_{k=1}^K$ is finite, and we do not include the $\sqrt{k/n}$ term as discussed in Remark 3.3, then we have that:

$$E_{P^n}[W_{\mathcal{G}}^* - W(\hat{G}_n)] \leq C_1 \sqrt{\frac{V_{k^*}}{n}} + \sqrt{\frac{\log(Kc_1e)}{2c_0n}},$$

where c_0, c_1 are as in Assumption 3.4.

This result shows that, if the distribution P is such that the optimum $W_{\mathcal{G}}^*$ is achieved in \mathcal{G}_{k^*} , then the upper bound on regret for PWM is as if we had performed EWM in \mathcal{G}_{k^*} even though we cannot know this class in practice. In Section 4 we will illustrate this result in a simulation study, in the setting of Example 3.1.

3.3 Penalized Welfare Maximization: Some Examples of Penalties

This section serves two purposes. First, it illustrates the results of Section 3.2 with two concrete choices for the penalty $C_n(k)$. Second, the results help quantify the size of the extraneous term in the bound of Theorem 3.1 for these penalties, so as to address the concerns presented in Remark 3.5. The first penalty we present, the Rademacher penalty, is theoretically elegant but computationally burdensome. The second penalty we present, the holdout penalty, is very intuitive and much more tractable in applications. However, the holdout penalty involves a sample-splitting procedure that some may find unappealing. Both of the penalties share the property that they do *not* require the practitioner to know the VC dimensions V_k of the approximating classes, which we feel is important to make the method broadly applicable.

3.3.1 The Rademacher Penalty

The first penalty we present is very attractive from a theoretical perspective, but is computationally burdensome. Let $S_n := \{(Y_i, D_i, X_i)\}_{i=1}^n$ be the observed data. Then the *Rademacher* penalty is given by

$$C_n(k) = E_{\sigma} \left[\sup_{G \in \mathcal{G}_k} \frac{2}{n} \sum_{i=1}^n \sigma_i \tau_i \mathbf{1}\{X_i \in G\} \mid S_n \right],$$

where τ_i is defined as in equation (2), and $\{\sigma_1, \dots, \sigma_n\}$ are a sequence of i.i.d Rademacher variables, i.e. they take on the values $\{-1, 1\}$, each with probability half.

To clarify the origin of this penalty, recall that $C_n(k)$ must be a good upper bound on $W_n(\hat{G}_{n,k}) - W(\hat{G}_{n,k})$, which is the requirement of Assumption 3.4. Bounding such quantities is common in the study of empirical processes, and the usual first step is to use what is known as *symmetrization*, which says that:

$$E_{P^n}[\sup_{G \in \mathcal{G}} W_n(G) - W(G)] \leq E_{P^n} \left[E_{\sigma} \left[\sup_{G \in \mathcal{G}} \frac{2}{n} \sum_{i=1}^n \sigma_i \tau_i \mathbf{1}\{X_i \in G\} \mid S_n \right] \right].$$

It is thus this inequality that inspires the definition of $C_n(k)$. The concept of Rademacher complexity⁵ is pervasive throughout the statistical learning literature (see for example [Koltchinskii \(2001\)](#), [Bartlett and Mendelson \(2002\)](#), and [Bartlett et al. \(2002\)](#)). Intuitively, it measures a notion of complexity that is finer than that of VC dimension, and is at the same time computable from the data at hand. Furthermore, unlike the holdout penalty introduced in the next subsection, it allows both the objective function and the penalty to be estimated with all of the data.

Our first task is to prove that the conditions of Assumptions [3.4](#) and [3.5](#) hold for the Rademacher penalty:

Lemma 3.1. *Consider Assumptions [2.1](#), [3.1](#), [3.3](#). Let $C_n(k)$ be the Rademacher penalty as defined above. Then we have that*

$$P^n(W_n(\hat{G}_{n,k}) - W(\hat{G}_{n,k}) - C_n(k) > \epsilon) \leq \exp\left(-2\left(\frac{\kappa}{3M}\right)^2 n \epsilon^2\right),$$

and

$$E_{P^n}[C_n(k)] \leq C \frac{M}{\kappa} \sqrt{\frac{V_k}{n}},$$

where C is the same universal constant that appears in equation [\(3\)](#).

We are thus able to refine Theorem [3.1](#) to the case of the Rademacher penalty.

Proposition 3.1. *Consider Assumptions [2.1](#), [3.1](#), [3.3](#). Let $C_n(k)$ be the Rademacher penalty as defined above. Then we have that for every $P \in \mathcal{P}(M, \kappa)$:*

$$E_{P^n}[W_{\mathcal{G}}^* - W(\hat{G}_n)] \leq \inf_k \left[E_{P^n}[C_n(k)] + (W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^*) + \sqrt{\frac{k}{n}} \right] + g(M, \kappa) \frac{M}{\kappa} \sqrt{\frac{1}{n}},$$

with $E_{P^n}[C_n(k)] \leq C \frac{M}{\kappa} \sqrt{\frac{V_k}{n}}$, where C is the same universal constant as that in equation [\(3\)](#) and

$$g(M, \kappa) := 6 \sqrt{\log\left(\frac{3\sqrt{e} M}{\sqrt{2} \kappa}\right)}.$$

Remark 3.6. We can now revisit the comment we made in Remark [3.5](#), about quantifying the size of the constants in the extraneous term of the bound. In Appendix [B](#) we perform a back-of-the-envelope calculation that provides insight into the size of $g(M, \kappa)$, and compares it to the size of the universal constant C derived in [Kitagawa and Tetenov \(2015\)](#). ■

Despite this penalty being theoretically appealing, implementing it in practical applications is problematic. The standard approach suggested in the statistical learning literature is to compute $C_n(k)$ by simulation: first, we repeatedly draw samples of $\{\sigma_i\}_{i=1}^n$, then we solve the problem

$$\max_{G \in \mathcal{G}_k} \frac{2}{n} \sum_{i=1}^n \sigma_i \tau_i \mathbf{1}\{X_i \in G\},$$

⁵Note that the definition of Rademacher complexity is slightly different than the definition of our penalty. Here we follow [Bartlett et al. \(2002\)](#) and do not include the absolute value in our definition of the penalty.

for each draw, and then average the result. Unfortunately, the optimization problem to be solved in the second step is computationally demanding for most classes \mathcal{G}_k of interest, so that repeatedly solving it for multiple draws of $\{\sigma_i\}_{i=1}^n$ is impractical. Moreover, this procedure must be repeated for *each* class \mathcal{G}_k , which makes it even more prohibitive.

In the next section, we present a penalty that is not only conceptually very simple, but easy to implement as well.

3.3.2 The Holdout Penalty

The second penalty we introduce is motivated by the following idea: First fix some number $\ell \in (0, 1)$ such that $m := n(1 - \ell)$ (for expositional clarity suppose this is an integer)⁶, and let $r := n - m$. Given our original sample $S_n = \{(Y_i, D_i, X_i)\}_{i=1}^n$, let $S_n^E := \{(Y_i, D_i, X_i)\}_{i=1}^m$ denote what we call the *estimating* sample, and let $S_n^T := \{(Y_i, D_i, X_i)\}_{i=m+1}^n$ denote the *testing* sample. Now, using S_n^E , compute $\hat{G}_{m,k}$ for each k . It seems intuitive that we could get a sense of the efficacy of $\hat{G}_{m,k}$ by applying this rule to the subsample S_n^T and computing the empirical welfare $W_r(\hat{G}_{m,k})$. We could then select the class k that results in the highest estimated welfare $W_r(\hat{G}_{m,k})$.

It turns out this idea can be formalized in our framework by treating it as a PWM-rule on the estimating sample, with the following penalty: for each EWM rule $\hat{G}_{m,k}$ estimated on S_n^E , let

$$W_m(\hat{G}_{m,k}) = \frac{1}{m} \sum_{i=1}^m \tau_i \mathbf{1}\{X_i \in \hat{G}_{m,k}\} ,$$

be the estimated welfare of the rule $\hat{G}_{m,k}$ on S_n^E and let

$$W_r(\hat{G}_{m,k}) = \frac{1}{r} \sum_{i=m+1}^n \tau_i \mathbf{1}\{X_i \in \hat{G}_{m,k}\} ,$$

be the estimated welfare of the rule $\hat{G}_{m,k}$ on S_n^T . We define the *holdout* penalty to be

$$C_m(k) := W_m(\hat{G}_{m,k}) - W_r(\hat{G}_{m,k}) .$$

Now, recall that the PWM rule is given by

$$\hat{G}_m = \arg \max_k [W_m(\hat{G}_{m,k}) - C_m(k) - \sqrt{\frac{k}{m}}] ,$$

which, given the definition of $C_m(k)$, simplifies to

$$\hat{G}_m = \arg \max_k [W_r(\hat{G}_{m,k}) - \sqrt{\frac{k}{m}}] .$$

Hence we see that the PWM rule with the holdout penalty reproduces the idea we presented above (with the usual addition of the $\sqrt{k/m}$ term; see Remark 3.3).

We check the conditions of Assumptions 3.4 and 3.5:

⁶The results would continue to hold if one were to instead define $m := \lfloor n(1 - \ell) \rfloor$.

Lemma 3.2. Assume Assumptions 2.1, 3.1, 3.3. Suppose we have a sample of size n and recall that $m = n(1 - \ell)$ and $r = n - m$. Let $C_m(k)$ be the holdout penalty as defined above. Then we have that

$$P^n(W_m(\hat{G}_{m,k}) - W(\hat{G}_{m,k}) - C_m(k) > \epsilon) \leq \exp\left(-2\left(\frac{\kappa}{M}\right)^2 n \ell \epsilon^2\right),$$

and

$$E_{P^n}[C_m(k)] \leq C \frac{M}{\kappa \sqrt{(1 - \ell)}} \sqrt{\frac{V_k}{n}},$$

where C is the same universal constant that appears in equation (3).

With Lemma 3.2 established, Theorem 3.1 becomes:

Proposition 3.2. Assume Assumptions 2.1, 3.1, 3.3. Suppose we have a sample of size n , and let $m = n(1 - \ell)$, $r = n - m$. Let $C_m(k)$ be the holdout penalty as defined above. Then we have that for every $P \in \mathcal{P}(M, \kappa)$:

$$E_{P^n}[W_{\mathcal{G}}^* - W(\hat{G}_m)] \leq \inf_k \left[E_{P^n}[C_n(k)] + (W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^*) + \sqrt{\frac{k}{n}} \right] + g(M, \kappa, \ell) \frac{M}{\kappa \sqrt{\ell}} \sqrt{\frac{1}{n}},$$

with

$$E_{P^n}[C_n(k)] \leq C \frac{M}{\kappa \sqrt{(1 - \ell)}} \sqrt{\frac{V_k}{n}},$$

where C is the same universal constant as that in equation (3) and

$$g(M, \kappa, \ell) := 2 \sqrt{\log\left(\sqrt{\frac{e}{2\ell}} \frac{M}{\kappa}\right)}.$$

Remark 3.7. We can perform the same analysis as we did in Remark 3.6. In doing so we see that the difference between this result and the result in Proposition 3.1 is that sample-splitting introduces distortions into the constant terms through ℓ . Indeed, the tradeoff between splitting the sample into the estimating sample and testing sample is reflected in these constants. ■

As noted in Remark 3.7, the bound we derive for the holdout penalty is similar to what we derive for the Rademacher penalty, but with inflated constants. However, the benefit of the holdout penalty lies in the fact that it is much more practical to implement. The only remaining issue with the holdout penalty is how to split the data. Deriving some sort of data-driven procedure to choose the proportion ℓ is beyond the scope of our paper, but as a rule of thumb, we have found that it is much more important to focus on accurate estimation of the rule $\hat{G}_{m,k}$ than on the computation of $W_r(\hat{G}_{m,k})$. In other words, we recommend that the estimating sample S_n^E be a large proportion of the original sample S_n . Throughout Sections 4 and 5, we designate three quarters of the sample as the estimating sample.

3.4 Penalized Welfare Maximization: Estimated Propensity Score

In this section we present a modification of the PWM rule where the propensity score is not known and must be estimated from the data. This situation would arise if the planner had access to observational data instead of data from a randomized experiment. Before describing our modification of the PWM rule, we must review results about the corresponding modification of the EWM rule in [Kitagawa and Tetenov \(2015\)](#). The modification we consider here is what they call the *e-hybrid* EWM rule. Recall the EWM objective function as defined in equation (2). To define the e-hybrid EWM rule we modify this objective function by replacing τ_i with

$$\hat{\tau}_i := \left[\frac{Y_i D_i}{\hat{e}(X_i)} - \frac{Y_i(1 - D_i)}{1 - \hat{e}(X_i)} \right] \mathbf{1}\{\epsilon_n \leq \hat{e}(X_i) \leq 1 - \epsilon_n\} ,$$

where $\hat{e}(\cdot)$ is an estimator of the propensity score, and ϵ_n is a trimming parameter such that $\epsilon_n = O(n^{-\alpha})$ for some $\alpha > 0$. Hence the e-hybrid EWM objective function is defined as follows:

$$W_n^e(G) := \frac{1}{n} \sum_{i=1}^n \hat{\tau}_i \mathbf{1}\{X_i \in G\} .$$

Since we are now estimating the propensity score, we must impose additional regularity conditions on P to guarantee a uniform rate of convergence. We make a high level assumption:

Assumption 3.7. *Given an estimator $\hat{e}(\cdot)$, let \mathcal{P}_e be a class of data generating processes such that*

$$\sup_{P \in \mathcal{P}_e} E_{P^n} \left[\frac{1}{n} \sum_{i=1}^n |\hat{\tau}_i - \tau_i| \right] = O(\phi_n^{-1}) ,$$

where $\phi_n \rightarrow \infty$.

Although we do not explore low-level conditions that satisfy this assumption here, [Kitagawa and Tetenov \(2015\)](#) do so in their paper. To summarize their results, they show that if $\hat{e}(\cdot)$ is a local polynomial estimator, and that $e(\cdot)$ and the marginal distribution of X satisfy some smoothness conditions, then Assumption 3.7 is satisfied with $\phi_n = n^{-\frac{1}{n+d_x/\beta_e}}$, where β_e is a constant that determines the smoothness of $e(\cdot)$.⁷

Let $\hat{G}_{e-hybrid}$ be the solution to the e-hybrid problem in a class \mathcal{G} of finite VC dimension, then [Kitagawa and Tetenov \(2015\)](#) derive the following bound on maximum \mathcal{G} -regret:

$$\sup_{P \in \mathcal{P}_e \cap \mathcal{P}(M, \kappa)} E_{P^n} \left[W_{\mathcal{G}}^* - W(\hat{G}_{e-hybrid}) \right] \leq O(\phi_n^{-1} \vee n^{-1/2}) . \quad (5)$$

With a non-parametric estimator of $e(\cdot)$, ϕ_n will generally be slower than \sqrt{n} and hence determine the rate of convergence.

⁷To be more precise, β_e is the degree of the Holder class to which $e(\cdot)$ must belong.

We are now ready to present the construction of the corresponding e-hybrid PWM estimator. Let \mathcal{G} be an arbitrary class of allocations, and let $\{\mathcal{G}_k\}_k$ be some approximating sequence for \mathcal{G} . Let $\hat{G}_{n,k}^e$ be the hybrid EWM rule in the class \mathcal{G}_k . Let $C_n^e(k)$ be our penalty for the hybrid PWM rule. We now require that the penalty satisfies the following properties:

Assumption 3.8. (*Assumptions on $C_n^e(k)$*)

In addition to making assumptions about $C_n^e(k)$, we assume there exists an “infeasible penalty” $\tilde{C}_n(k)$ with the following properties:

- *There exist positive constants c_0 and c_1 such that $\tilde{C}_n(k)$ satisfies the following tail inequality for every n, k and for every $\epsilon > 0$:*

$$\sup_{P \in \mathcal{P}_e \cap \mathcal{P}(M, \kappa)} P^n(W_n(\hat{G}_{n,k}^e) - W(\hat{G}_{n,k}^e) - \tilde{C}_n(k) > \epsilon) \leq c_1 e^{-2c_0 n \epsilon^2}$$

- *There exists a positive constant C_1 such that, for every n , $\tilde{C}_n(k)$ satisfies*

$$\sup_{P \in \mathcal{P}_e \cap \mathcal{P}(M, \kappa)} E_{P^n}[\tilde{C}_n(k)] \leq C_1 \sqrt{\frac{V_k}{n}},$$

where V_k is the VC dimension of \mathcal{G}_k .

- *$\tilde{C}_n(k)$ and $C_n^e(k)$ are such that*

$$\sup_{P \in \mathcal{P}_e \cap \mathcal{P}(M, \kappa)} E_{P^n} \left[\sup_k \left| C_n^e(k) - \tilde{C}_n(k) \right| \right] = O(\phi_n^{-1}).$$

We will provide context for these assumptions. First of all, included in the assumptions on $C_n^e(k)$ is the existence of an object $\tilde{C}_n(k)$ which we call an *infeasible penalty*. The first assumption asserts that the infeasible penalty obeys a similar tail inequality to $C_n(k)$, which was the penalty when the propensity score was known. The main difference is that $\tilde{C}_n(k)$ satisfies this assumption with respect to the *e-hybrid* EWM rule, and not the EWM rule with a known propensity. What is strange about this condition is that it is as if we were evaluating the hybrid rule through the empirical objective $W_n(\cdot)$, *which is the objective when the propensity score is known*. This is our motivation for calling $\tilde{C}_n(k)$ an infeasible penalty. Luckily, $\tilde{C}_n(k)$ is purely a theoretical device and does not serve a role in the actual implementation of PWM. We provide an example of such an infeasible penalty in the setting of the holdout penalty below.

The second assumption is the same as Assumption 3.5, but now with respect to the infeasible penalty $\tilde{C}_n(k)$. The third assumption simply links the true penalty $C_n^e(k)$ to the infeasible penalty $\tilde{C}_n(k)$ in such a way that both should agree asymptotically and do so at an appropriate rate.

Given this, we obtain the following analogue to Theorem 3.1:

Theorem 3.2. *Given assumptions 2.1, 3.1, 3.3, 3.7 and 3.8, there exist constants Δ and c_0 such that for every $P \in \mathcal{P}_e \cap \mathcal{P}(M, \kappa)$:*

$$E_{P^n}[W_{\mathcal{G}}^* - W(\hat{G}_n^e)] \leq \inf_k \left[E_{P^n}[\tilde{C}_n(k)] + (W_{\mathcal{G}}^* - W_{\hat{\mathcal{G}}_k}^*) + \sqrt{\frac{k}{n}} \right] + O(\phi_n^{-1}) + \sqrt{\frac{\log(\Delta e)}{2c_0 n}} .$$

As we can see, the only difference between this bound and the bound derived in Theorem 3.1 is that there is an additional term of order ϕ_n^{-1} . This is also the case with the hybrid EWM estimator, as shown in (5).

Next, we check that the conditions in Assumption 3.8 are satisfied with modified versions of the holdout and Rademacher penalties. First let's begin with the holdout penalty. Recall from Section 3.3 that the holdout method split the sample $S_n = \{(Y_i, D_i, X_i)\}_{i=1}^n$ into the estimating sample $S_n^E = \{(Y_i, D_i, X_i)\}_{i=1}^m$ of size $m = n(1 - \ell)$ and the testing sample $S_n^T = \{(Y_i, D_i, X_i)\}_{i=m+1}^n$ of size $r = n - m$. The holdout penalty was then defined as

$$C_m(k) = W_m(\hat{G}_{m,k}) - W_r(\hat{G}_{m,k}) ,$$

where $W_m(\cdot)$ was the empirical welfare computed on S_n^E and $W_r(\cdot)$ was the empirical welfare computed on S_n^T .

To define the *hybrid holdout* penalty, let $\hat{e}^E(\cdot)$ be the propensity estimated on S_n^E , and let $\hat{e}^T(\cdot)$ be the propensity estimated on S_n^T . Define

$$W_m^e(G) := \frac{1}{m} \sum_{i=1}^m \hat{\tau}_i^E \mathbf{1}\{X_i \in G\} ,$$

where

$$\hat{\tau}_i^E = \left[\frac{Y_i D_i}{\hat{e}^E(X_i)} - \frac{Y_i(1 - D_i)}{1 - \hat{e}^E(X_i)} \right] \mathbf{1}\{\epsilon_n \leq \hat{e}^E(X_i) \leq 1 - \epsilon_n\} .$$

Define $W_r^e(G)$ on the testing sample analogously. Letting $\hat{G}_{m,k}^e$ be the hybrid EWM rule computed on the estimating sample in the class \mathcal{G}_k , the hybrid holdout penalty is defined as:

$$C_m^e(k) := W_m^e(\hat{G}_{m,k}^e) - W_r^e(\hat{G}_{m,k}^e) .$$

We can now check the conditions of Assumption 3.8 for the hybrid holdout penalty. To do so, we must assert the existence of an infeasible penalty $\tilde{C}_m(k)$ that satisfies our assumptions. The infeasible penalty we consider is given by

$$\tilde{C}_m(k) := W_m(\hat{G}_{m,k}^e) - W_r(\hat{G}_{m,k}^e) ,$$

where $W_m(\cdot)$ and $W_r(\cdot)$ are defined as in Section 3.3, that is, they are computed *as if the propensity was known*. We present the following lemma:

Lemma 3.3. Assume Assumptions 2.1, 3.1, 3.3, and 3.7. Suppose we have a sample of size n and recall that $m = n(1 - \ell)$ and $r = n - m$. Let $C_m^e(k)$ be the hybrid holdout penalty and $\tilde{C}_m(k)$ be the infeasible penalty as defined above. Then we have that

$$P^n(W_m(\hat{G}_{m,k}^e) - W(\hat{G}_{m,k}^e) - \tilde{C}_m(k) > \epsilon) \leq \exp\left(-2\left(\frac{\kappa}{M}\right)^2 n \ell \epsilon^2\right),$$

$$E_{P^n}[\tilde{C}_m(k)] \leq C \frac{M}{\kappa \sqrt{(1 - \ell)}} \sqrt{\frac{V_k}{n}},$$

and

$$\sup_{P \in \mathcal{P}_e} E_{P^n} \left[\sup_k |C_m^e(k) - \tilde{C}_m(k)| \right] = O(\phi_n^{-1}),$$

where C is the same universal constant as that in equation (3).

We thus obtain an analogous result to Proposition 3.2 for PWM with the hybrid holdout penalty. Next we do the same thing for the Rademacher penalty. In fact, defining the hybrid version of the Rademacher penalty is relatively straightforward. Recall that the Rademacher penalty when the propensity score was known was defined as

$$C_n(k) = E_\sigma \left[\sup_{G \in \mathcal{G}_k} \frac{2}{n} \sum_{i=1}^n \sigma_i \tau_i \mathbf{1}\{X_i \in G\} \mid S_n \right].$$

The *hybrid* Rademacher penalty is defined analogously:

$$C_n^e(k) = E_\sigma \left[\sup_{G \in \mathcal{G}_k} \frac{2}{n} \sum_{i=1}^n \sigma_i \hat{\tau}_i \mathbf{1}\{X_i \in G\} \mid S_n \right].$$

To check the conditions of Assumption 3.8, the infeasible penalty we consider here is simply just the penalty when the propensity score is known, so that $\tilde{C}_n(k) = C_n(k)$. Hence we have the following lemma:

Lemma 3.4. Assume Assumptions 2.1, 3.1, 3.3, and 3.7. Let $C_n^e(k)$ be the hybrid Rademacher penalty and $\tilde{C}_n(k)$ be the infeasible penalty as defined above. Then we have that

$$P^n(W_n(\hat{G}_{n,k}^e) - W(\hat{G}_{n,k}^e) - \tilde{C}_n(k) > \epsilon) \leq \exp\left(-2\left(\frac{\kappa}{3M}\right)^2 n \epsilon^2\right),$$

$$E_{P^n}[\tilde{C}_n(k)] \leq C \frac{M}{\kappa} \sqrt{\frac{V_k}{n}},$$

and

$$\sup_{P \in \mathcal{P}_e} E_{P^n} \left[\sup_k |C_n^e(k) - \tilde{C}_n(k)| \right] = O(\phi_n^{-1}),$$

where C is the same universal constant as that in equation (3).

Again, from this we obtain an analogous result to Proposition 3.1 for PWM with the hybrid Rademacher penalty.

4 A Simulation Study

In this section we perform a small simulation study to highlight the ability of the PWM rule to reduce pointwise \mathcal{G} -regret in an empirically relevant setting. We consider a situation where the planner has access to threshold-type allocations, of the type described in Examples 2.2 and 3.1, but with five available covariates. The sieve sequence we consider is the same as in Example 3.1, where \mathcal{G}_k is the set of threshold allocations on $k - 1$ out of the 5 covariates. For example, \mathcal{G}_1 contains only the allocations $G = \emptyset$ and $G = \mathcal{X}$, which correspond to threshold allocations that use zero covariates, \mathcal{G}_2 contains all threshold allocations on one out of the five covariates, etc.

The problem that the planner faces is choosing how many covariates to use in the allocation: for example suppose that the distribution P is such that some of the available covariates are irrelevant for assigning treatment. Of course, the planner could perform EWM on all the covariates at once, and by the bound in equation (3) this is guaranteed to produce small regret in large enough samples. However, if the sample is not large, the planner may be able to achieve a reduction in regret by performing PWM. Through the lens of Corollary 3.2, our results say that PWM should behave *as if* we had performed EWM in the smallest class \mathcal{G}_k that contains all of the relevant covariates.

To be concrete, we consider the following data generating process: Let $\mathcal{X} = [0, 1]^5$, and

$$X_i = (X_{1i}, X_{2i}, \dots, X_{5i}) \sim (U[0, 1])^5 .$$

The potential outcomes for unit i are specified as:

$$Y_i(1) = 50(2X_{2i} - (1 - X_{1i})^4 - 0.5 + 0.5(X_{3i} - X_{4i})) + U_{1i} ,$$

$$Y_i(0) = 50(0.5(X_{3i} - X_{4i})) + U_{2i} ,$$

where U_1 and U_2 are distributed as $U[-20, 20]$ random variables which are independent of each other and of X . The covariates enter the potential outcomes in three different ways:

- X_{5i} is an irrelevant covariate; it does not play a role in determining potential outcomes at all.
- X_{3i} and X_{4i} affect both treatment and control equally; there will be a nonzero correlation between the observed outcome Y_i and these covariates, but they serve no purpose for treatment assignment.
- X_{1i} and X_{2i} *do* serve a purpose for assigning treatment, and both are used in the optimal allocation. See Figure 1 below.

[Figure 1 about here.]

To implement PWM we used the holdout penalty, with 3/4 of our sample designated as the estimating sample. In Appendix C we explain in detail how we implemented PWM as a series of mixed integer linear programs, and how we performed our simulations. Although the method we present is brute-force, we have found that our implementation should be feasible on a personal computer with as many as eight covariates in practice. Our results compare the \mathcal{G} -regret of the PWM rule against the regret of performing EWM in \mathcal{G}_6 (which corresponds to the class that uses all five covariates) or performing EWM in \mathcal{G}_3 . Recall that \mathcal{G}_3 is the smallest class that contains the optimal allocation. In light of Corollary 3.2, we would hope that PWM behaves similarly to doing EWM in \mathcal{G}_3 directly. In Figure 2, we plot the regret of these rules for various sample sizes.

[Figure 2 about here.]

First we comment on the regret of performing EWM in \mathcal{G}_6 (recall that this corresponds to the set of allocations using all five covariates) vs. performing EWM in \mathcal{G}_3 (which corresponds to the set of allocations that use two of the five covariates). As predicted by equation (3), regret decreases as sample size increases. Moreover, performing EWM in \mathcal{G}_6 results in larger regret at every sample size: performing EWM in \mathcal{G}_3 results in a 6% improvement on average, across the sample sizes we consider.

Next, we comment on the performance of PWM. As we had hoped, the regret of PWM is smaller than the regret of performing EWM in \mathcal{G}_6 at every sample size: performing PWM results in a 4% improvement on average, across the sample sizes we consider. Moreover, the results in Figure 2 suggest that this gain is not just due to an improvement in very small samples, as the gap in regret seems to diminish quite slowly as sample size increases.

5 An Application

In this section we apply the PWM rule to experimental data from the Job Training Partnership Act (JTPA) Study. The JTPA study was a randomized controlled trial whose purpose was to measure the benefits and costs of employment and training programs. The study randomized whether applicants would be eligible to receive a collection of services provided by the JTPA related to job training, for a period of 18 months. The study collected background information about the applicants prior to the experiment, as well as data on applicants' earnings for 30 months following assignment (for a detailed description of the study, see Bloom et al. (1997)).⁸

⁸The sample we use is the same as that in Abadie et al. (2013), which we downloaded from ideas.repec.org/c/boc/bocode/s457801.html. We supplemented this dataset with education data from the `expbif.dta` dataset available at the W.E. Upjohn Institute website. Observations with years of education coded as '99' were dropped.

We revisit the application setting of [Kitagawa and Tetenov \(2015\)](#). The outcome that we consider is total individual earnings in the 30 months following program assignment. The covariates on which we define our treatment allocations are the individual’s years of education and their earnings in the year prior to the assignment. The set of allocations we consider is the set of monotone allocations defined in [Example 2.3](#), but with a *non-increasing* monotone function. To be precise, let \mathcal{X}_1 be the covariate set of years of education, and let \mathcal{X}_2 be the covariate set of previous earnings, then the set of allocations we consider is given by:

$$\mathcal{G} = \{G : G = \{(x_1, x_2) \in \mathcal{X} \mid x_2 \leq f(x_1) \text{ for } f : \mathcal{X}_1 \rightarrow \mathcal{X}_2 \text{ non-increasing}\}\} .$$

Let us discuss what this set of allocations means in the context of this application. This restriction imposes that, the less education you have, the more accessible is the program based on your previous earnings. For example, if an applicant with 12 years of education and previous earnings of \$20,000 is to be accepted into the program, then an applicant with the same previous earnings and less education must also be accepted, as well as an applicant with the same level of education and less earnings. In [Example 2.3](#) we discussed a situation where application-specific assumptions impose this type of constraint. In this setting, we instead argue that it is plausible that such a restriction may be exogenously imposed on the planner for political reasons; after all, it may not be politically viable to implement a job-training program where only those with high levels of education are accepted.

As we have previously discussed, this class of allocations will have infinite VC dimension when continuous covariates are used. Accordingly, in the results that follow, we will assume both covariates are continuous. However, note that in our application years of education is a discrete covariate. This discrepancy is not an issue for illustrating our method, and we think it is important that we make our study comparable to the one in [Kitagawa and Tetenov \(2015\)](#).

The approximating sequence we consider is the one described in [Example 3.2](#), but now with a non-increasing monotonicity constraint. Recall that this was a sequence such that the resulting allocations partitioned the covariate space with a progressively refined, piecewise-linear, monotone boundary. Given any fixed class in this sequence, we can perform EWM in that class. For example, [Figure 3](#) below illustrates the result of performing EWM on the simplest class in the approximating sequence. This class is equivalent to the class of linear treatment rules from [Kitagawa and Tetenov \(2015\)](#), but with an additional slope constraint.

[Figure 3 about here.]

At the other end of the spectrum, we could consider performing EWM in the most complicated class in our approximating sequence: this class corresponds to allocations that stipulate a threshold for previous income at every level of education (note that such a class exists here because years of education is discrete). [Figure 4](#) below illustrates the result of performing EWM in this class.

[Figure 4 about here.]

As we would expect, the resulting allocation in the simplest class and in the most complicated class are quite different, and given the option to choose any class from our sequence, it is not obvious which one should be chosen given the size of the experiment. As we have seen throughout this paper, PWM will select the appropriate class to use. Before showing the results for the PWM rule, recall from Remark 3.1 that, if the class \mathcal{G} has infinite VC dimension (as it would if both covariates were continuous), then we *cannot* establish a bound on maximum regret without imposing additional regularity conditions. Accordingly, we will first establish a set of regularity conditions under which we derive a bound on maximum regret of the PWM rule.

We state the result for $\mathcal{X} = [0, 1]^2$. We impose the following regularity condition on the distribution P :

Assumption 5.1. *Let \mathcal{P}_r be a set of distributions such that there exists some constant A , where for every $P \in \mathcal{P}_r$, X has a density p_x with respect to Lebesgue measure on $[0, 1]^2$ such that p_x is bounded above by A .*

It is worth emphasizing that we do *not* require the first best to be contained in \mathcal{G} , nor do we require that $W_{\mathcal{G}}^*$ even be attained. With this regularity condition imposed, we are able to derive the following uniform bound on the approximation bias $W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^*$:

Proposition 5.1. *Under Assumption 5.1, the approximation bias of the approximating sequence $\{\mathcal{G}_k\}_{k=1}^{\infty}$ from Example 3.2 satisfies*

$$\sup_{P \in \mathcal{P}_r} W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^* \leq A \frac{M}{\kappa} 2^{-k} ,$$

To illustrate the use of Proposition 5.1 in our setting, we derive a bound on maximum regret for monotone allocations. Proposition 5.1 and Corollary 3.1, along with the (possibly loose) bound on V_k given in Example 3.2 allow us to conclude that:

Corollary 5.1. *Let $C_n(k)$ be the Rademacher or holdout penalty. Under Assumptions 2.1, 3.1, 3.3, and 5.1, we have that*

$$\sup_{P \in \mathcal{P}_r \cap \mathcal{P}(M, \kappa)} E_{P^n} [W_{\mathcal{G}}^* - W(\hat{G}_n)] = O(n^{-\frac{1}{3}}) .$$

As we alluded to in the discussion of Example 3.2, bounds on maximum regret for EWM can be derived for the class of monotone allocations. It is interesting to compare our bound to the bound derived using the proof presented in Györfi et al. (1996) in the context of classification:

Proposition 5.2. *Under Assumptions 2.1, 3.1, and 5.1, we have that*

$$\sup_{P \in \mathcal{P}_r \cap \mathcal{P}(M, \kappa)} E_{P^n} [W_{\mathcal{G}}^* - W(\hat{G}_{EWM})] = O(n^{-\frac{1}{4}}) .$$

We make no claim that either of these bounds are sharp: for EWM, the most relevant results of which we are aware are presented in [Tsybakov \(2004\)](#), where he shows that if the optimum is achieved in \mathcal{G} , and sufficient smoothness is imposed on the boundary of the optimal allocation, then the rate of convergence of the classification analogue of EWM is $O(n^{-1/2})$. Another relevant result from classification comes from [Tsybakov and van de Geer \(2005\)](#), where they develop a penalized method for classification over boundary fragments which is able to achieve a root-n rate (up to a logarithmic factor) for monotone allocations, while only assuming that the optimum is achieved. It would be interesting to understand to what extent these techniques generalize to our setting, and also whether or not PWM is truly able to achieve a faster rate of convergence over EWM for this example under our assumptions.

In [Figure 5](#), we illustrate the result of performing PWM on our sequence of classes, where we used 3/4 of our sample for estimation. In [Appendix C](#) we discuss the computational details of our implementation. Note that PWM selects the allocation from the second class in our sequence, which corresponds to a piecewise-linear allocation with one allowable “kink”.

[Figure 5 about here.]

6 Conclusion

In this paper, we introduced a new statistical decision rule for the treatment assignment problem, which we call the Penalized Welfare Maximization (PWM) rule. Our rule builds on the Empirical Welfare Maximization Rule of [Kitagawa and Tetenov \(2015\)](#), which is designed for situations where treatment allocation is exogenously constrained, in two ways. First, we showed that our rule is well suited for deriving bounds on maximum regret for a broad set of classes of infinite VC dimension. We illustrated this feature of PWM by deriving bounds on maximum regret when the class of treatment allocations are monotone allocations, and applied this class of allocations to the JTPA study. Second, our rule is also able to reduce regret in applications where sample size is small relative to the complexity of the class of treatment allocations. We illustrated this feature of PWM in a simulation study where the planner has many covariates over which to define threshold allocations, but does not know which (or how many) of these covariates to use in practice.

Moving forward, we have identified some areas that we feel are worth further study. In general, implementing PWM is computationally challenging; from a practical perspective, practitioners may find it convenient to have a software package that can implement PWM in a few important examples. As we hinted at in [Example 3.3](#), trees are becoming popular for the estimation of treatment effects, and as we illustrate in [Appendix B](#) could serve as a useful approximating classes in our setting. We hope to further study the use of decision trees in the treatment assignment problem, as well as implement a software package that implements decision-tree based rules for practitioners.

A Proofs of Main Results

Recall that the planner's objective function is given by

$$W(G) = E_P \left[\left(\frac{YD}{e(X)} - \frac{Y(1-D)}{1-e(X)} \right) \cdot \mathbf{1}\{X \in G\} \right] . \quad (6)$$

To each treatment allocation $G \in \mathcal{G}$ we associate a function $f_G : \mathbb{R} \times \mathcal{X} \times \{0, 1\} \rightarrow \mathbb{R}$ defined by:

$$f_G(Z) = f_G(Y, X, D) = \left(\frac{YD}{e(X)} - \frac{Y(1-D)}{1-e(X)} \right) \cdot \mathbf{1}\{X \in G\} ,$$

where $Z = (Y, X, D)$. Let $\mathcal{F} := \{f_G : G \in \mathcal{G}\}$ denote the corresponding set of functions associated to decision rules in \mathcal{G} . By (6), any optimal allocation in \mathcal{G} solves

$$G^* \in \arg \max_{G \in \mathcal{G}} E_P \left[\left(\frac{YD}{e(X)} - \frac{Y(1-D)}{1-e(X)} \right) \cdot \mathbf{1}\{X \in G\} \right] .$$

Equivalently, functions associated to optimal allocations solve

$$f^* \in \arg \max_{f \in \mathcal{F}} E_P f(Z) .$$

By an abuse of notation, for $G \in \mathcal{G}$, we set

$$W(f_G) = E_P f_G(Z) .$$

Given an approximating sequence $\{\mathcal{G}_k\}_k$ of classes of treatment allocations, let $\{\mathcal{F}_k\}_k$ denote the sequence of associated classes of functions.

The following lemma, whose proof is given in [Kitagawa and Tetenov \(2015\)](#) (Lemma A.1), establishes the relevant link between the classes of sets $\{\mathcal{G}_k\}_k$ and the classes of functions $\{\mathcal{F}_k\}_k$. It shows that if a class \mathcal{G} has finite VC dimension, then the associated class \mathcal{F} is a VC-subgraph class with dimension bounded above by that of \mathcal{G} .

Lemma A.1. *Let \mathcal{G} be a VC-class of subsets of \mathcal{X} with finite VC dimension V . Let g be a function from $\mathcal{Z} := \mathbb{R} \times \mathcal{X} \times \{0, 1\}$ to \mathbb{R} . Then the set of functions \mathcal{F} defined by*

$$\mathcal{F} = \{g(z) \cdot \mathbf{1}\{x \in G\} : G \in \mathcal{G}\}$$

is a VC-subgraph class with dimension at most V .

For each $k \geq 1$, let $\hat{f}_{n,k}$ be a maximizer of the empirical welfare over the class \mathcal{F}_k ; that is:

$$\hat{f}_{n,k} = \arg \max_{f \in \mathcal{F}_k} W_n(f) ,$$

and for $f \in \mathcal{F}_k$, define the complexity-penalized estimate of welfare by

$$R_{n,k}(f) = W_n(f) - C_n(k) - \sqrt{\frac{k}{n}} .$$

The PWM rule $\hat{f}_{n,\hat{k}}$ is then chosen such that

$$\hat{k} = \arg \max_{k \geq 1} R_{n,k}(\hat{f}_{n,k}) .$$

In what follows, we set $\hat{f}_n := \hat{f}_{n,\hat{k}}$ and $R_n(\hat{f}_n) := R_{n,\hat{k}}(\hat{f}_{n,\hat{k}})$.

To bound the regret, we decompose it as follows

$$W_{\mathcal{F}}^* - W(\hat{f}_n) = \left(W_{\mathcal{F}}^* - R_n(\hat{f}_n) \right) + \left(R_n(\hat{f}_n) - W(\hat{f}_n) \right) . \quad (7)$$

The following lemma yields (under Assumption 3.4) a subgaussian tail bound for the second term on the right hand side of the preceding equality.

Lemma A.2. *Given Assumption 3.4, there exists a positive constant Δ (that does not depend on n) such that:*

$$P(R_n(\hat{f}_n) - W(\hat{f}_n) > \epsilon) \leq \Delta e^{-2c_o n \epsilon^2}$$

for every n .

Proof. First note that:

$$P(R_n(\hat{f}_n) - W(\hat{f}_n) > \epsilon) \leq P\left(\sup_k (R_{n,k}(\hat{f}_{n,k}) - W(\hat{f}_{n,k})) > \epsilon\right) ,$$

then by the union bound:

$$P\left(\sup_k (R_{n,k}(\hat{f}_{n,k}) - W(\hat{f}_{n,k})) > \epsilon\right) \leq \sum_k P(R_{n,k}(\hat{f}_{n,k}) - W(\hat{f}_{n,k}) > \epsilon) .$$

Now by definition of $R_{n,k}$, we have

$$\sum_k P(R_{n,k}(\hat{f}_{n,k}) - W(\hat{f}_{n,k}) > \epsilon) = \sum_k P(W_n(\hat{f}_{n,k}) - C_n(k) - W(\hat{f}_{n,k}) > \epsilon + \sqrt{\frac{k}{n}}) .$$

By Assumption 3.4,

$$\sum_k P(W_n(\hat{f}_{n,k}) - W(\hat{f}_{n,k}) - C_n(k) > \epsilon + \sqrt{\frac{k}{n}}) \leq \sum_k c_1 e^{-2c_o n (\epsilon + \sqrt{\frac{k}{n}})^2} \leq e^{-2c_o n \epsilon^2} \sum_k c_1 e^{-2kc_o} .$$

By setting

$$\Delta := \sum_k c_1 e^{-2kc_o} < \infty , \quad (8)$$

the result follows. ■

Proof of Theorem 3.1. We follow the general strategy from Bartlett et al. (2002). For every k , we have

$$W_{\mathcal{F}}^* - W(\hat{f}_n) = (W_{\mathcal{F}}^* - W_{\mathcal{F}_k}^*) + (W_{\mathcal{F}_k}^* - W(\hat{f}_n)) . \quad (9)$$

We first consider the second term in (9), and expand it as follows

$$W_{\mathcal{F}_k}^* - W(\hat{f}_n) = W_{\mathcal{F}_k}^* - R_n(\hat{f}_n) + R_n(\hat{f}_n) - W(\hat{f}_n) . \quad (10)$$

By the definition of R_n , the first term of expression (10) is bounded by

$$W_{\mathcal{F}_k}^* - R_n(\hat{f}_n) \leq W_{\mathcal{F}_k}^* - W_n(\hat{f}_{n,k}) + C_n(k) + \sqrt{\frac{k}{n}} .$$

Fix $\delta > 0$, and choose some $f_k^* \in \mathcal{F}_k$ such that $W(f_k^*) + \delta \geq W_{\mathcal{F}_k}^*$.⁹ We have

$$W_{\mathcal{F}_k}^* - W_n(\hat{f}_{n,k}) + C_n(k) + \sqrt{\frac{k}{n}} \leq W(f_k^*) + \delta - W_n(f_k^*) + C_n(k) + \sqrt{\frac{k}{n}} .$$

Taking expectations of both sides and letting δ converge to 0 yields

$$E[W_{\mathcal{F}_k}^* - R_n(\hat{f}_n)] \leq E[C_n(k)] + \sqrt{\frac{k}{n}} .$$

By Lemma A.2 and a standard integration argument (see for instance problem 12.1 in Györfi et al. (1996)), the second term on the right hand side of (10) is bounded by

$$E[R_n(\hat{f}_n) - W(\hat{f}_n)] \leq \sqrt{\frac{\log(\Delta e)}{2c_o n}} .$$

Combining these bounds yields

$$E[W_{\mathcal{F}}^* - W(\hat{f}_n)] \leq E[C_n(k)] + W_{\mathcal{F}}^* - W_{\mathcal{F}_k}^* + \sqrt{\frac{\log(\Delta e)}{2c_o n}} + \sqrt{\frac{k}{n}} ,$$

for every k , and our result follows. ■

Proof of Lemma 3.1. We first establish the inequality

$$P(W_n(\hat{f}_{n,k}) - W(\hat{f}_{n,k}) - C_n(k) > \epsilon) \leq \exp\left(-2n\epsilon^2\left(\frac{\kappa}{3M}\right)^2\right) . \quad (11)$$

By two standard symmetrization arguments, we get

$$E\left[\sup_{f \in \mathcal{F}_k} W_n(f) - W(f)\right] \leq 2E\left[\sup_{f \in \mathcal{F}_k} \frac{1}{n} \sum_{i=1}^n \sigma_i f(Z_i)\right] = E[C_n(k)] , \quad (12)$$

where we recall that $C_n(k) = E\left[2 \sup_{f \in \mathcal{F}_k} \frac{1}{n} \sum_{i=1}^n \sigma_i f(Z_i) | Z_1, Z_2, \dots, Z_n\right]$ and $\{\sigma_i\}_{i=1}^n$ is an *i.i.d* sequence of Rademacher random variables independent from the data $\{Z_i\}_{i=1}^n$. Note that

$$P(W_n(\hat{f}_{n,k}) - W(\hat{f}_{n,k}) - C_n(k) > \epsilon) \leq P\left(\sup_{f \in \mathcal{F}_k} ((W_n(f) - W(f)) - C_n(k)) > \epsilon\right) ,$$

⁹If the welfare criterion achieves its maximum on \mathcal{F}_k , then f_k^* can be set equal to any maximizer. In general however such an optimum may not exist, and thus we must choose f_k^* will to be an "almost maximizer" of the welfare criterion on \mathcal{F}_k .

and set $M_{n,k} := \sup_{f \in \mathcal{F}_k} (W_n(f) - W(f)) - C_n(k)$. Combining the preceding inequality with (12) yields

$$P(W_n(\hat{f}_{n,k}) - W(\hat{f}_{n,k}) - C_n(k) > \epsilon) \leq P(M_{n,k} - EM_{n,k} > \epsilon) .$$

To control the deviations of $M_{n,k}$ from its mean, we use McDiarmid's inequality (note that $M_{n,k}$ satisfies the bounded difference property with increments bounded by $\frac{3M}{n\kappa}$) which yields the inequality

$$P(M_{n,k} - EM_{n,k} > \epsilon) \leq \exp\left(-2n\epsilon^2\left(\frac{\kappa}{3M}\right)^2\right) ,$$

from which our result follows.

The second inequality (where C is a universal constant)

$$E[C_n(k)] \leq C \frac{M}{\kappa} \sqrt{\frac{V_k}{n}} ,$$

follows from a chaining argument and a control on the universal entropy of VC subgraph classes (see for instance the proof of Lemma A.4 in [Kitagawa and Tetenov \(2015\)](#)), along with Lemma A.1. ■

Proof of Lemma 3.2. Let us assume for notational simplicity that the quantity $m = n(1 - \ell)$ is an integer. We first establish the inequality

$$P(W_m(\hat{f}_{m,k}) - W(\hat{f}_{m,k}) - C_m(k) > \epsilon) \leq \exp\left(-2n\ell\epsilon^2\left(\frac{\kappa}{M}\right)^2\right) . \quad (13)$$

By the definition of $C_m(k)$, we have

$$P(W(\hat{f}_{m,k}) - W(\hat{f}_{m,k}) - C_m(k) > \epsilon) = P(W_r(\hat{f}_{m,k}) - W(\hat{f}_{m,k}) > \epsilon) .$$

Now, working conditionally on $\{Z_i\}_{i=1}^m$, we get by Hoeffding's inequality that

$$P(W_r(\hat{f}_{m,k}) - W(\hat{f}_{m,k}) > \epsilon | \{Z_i\}_{i=1}^m) \leq \exp\left(-2n\ell\epsilon^2\left(\frac{\kappa}{M}\right)^2\right) .$$

Since the right hand side of the preceding inequality is non random, the inequality holds unconditionally as well.

We now establish the inequality

$$E[C_m(k)] \leq C \frac{M}{\kappa\sqrt{1-\ell}} \sqrt{\frac{V_k}{n}} .$$

By the definition of $C_m(k)$, we have

$$E[C_m(k)] = E[W_m(\hat{f}_{m,k}) - W_r(\hat{f}_{m,k})] = E[W_m(\hat{f}_{m,k}) - W(\hat{f}_{m,k}) + W(\hat{f}_{m,k}) - W_r(\hat{f}_{m,k})] .$$

Note that by the law of iterated expectations, we have

$$E[W(\hat{f}_{m,k}) - W_r(\hat{f}_{m,k})] = 0 ,$$

and by Lemma A.4 in [Kitagawa and Tetenov \(2015\)](#) combined with Lemma [A.1](#) there exists some universal constant C such that:

$$E[W_m(\hat{f}_{m,k}) - W(\hat{f}_{m,k})] \leq C \frac{M}{\kappa} \sqrt{\frac{V_k}{m}}.$$

Since $m = (1 - \ell)n$, the result follows. ■

Proof of Propositions [3.1](#) and [3.2](#). From the inequality

$$\frac{e^{-x}}{(1 - e^{-x})} \leq \frac{1}{x},$$

and from [\(8\)](#) and [\(11\)](#), we derive that

$$\Delta \leq 1/2 \left(\frac{3M}{\kappa} \right)^2.$$

Similarly, we derive from [\(8\)](#) and [\(13\)](#) that

$$\Delta \leq 1/(2l) \left(\frac{M}{\kappa} \right)^2.$$

The results then follow by substituting these into the inequalities of Theorem [3.1](#). ■

Proof of Theorem [3.2](#). Our strategy here is to proceed analogously to the proof of Theorem [3.1](#) with some additional machinery. For every k , we have that:

$$W_{\mathcal{F}}^* - W(\hat{f}_n^e) = (W_{\mathcal{F}}^* - W_{\mathcal{F}_k}^*) + (W_{\mathcal{F}_k}^* - W(\hat{f}_n^e)). \quad (14)$$

Adding and subtracting $R_n^e(\hat{f}_n^e)$ to the last term yields

$$W_{\mathcal{F}_k}^* - W(\hat{f}_n^e) = (W_{\mathcal{F}_k}^* - R_n^e(\hat{f}_n^e)) + (R_n^e(\hat{f}_n^e) - W(\hat{f}_n^e)). \quad (15)$$

Let $f_k^* := \arg \max_{f \in \mathcal{F}_k} W(f)$, (if the supremum is not achieved, apply the argument to a δ -maximizer of the welfare, and let δ tend to zero). Now consider the first term on the right hand side of [\(15\)](#). Expanding yet again gives

$$W_{\mathcal{F}_k}^* - R_n^e(\hat{f}_n^e) = W_{\mathcal{F}_k}^* - W_n(f_k^*) + W_n(f_k^*) - R_n^e(\hat{f}_n^e). \quad (16)$$

From the definition of R_n^e , we have

$$W_n(f_k^*) - R_n^e(\hat{f}_n^e) \leq W_n(f_k^*) - W_n^e(f_k^*) + C_n^e(k) + \sqrt{\frac{k}{n}} \leq \frac{1}{n} \sum_{i=1}^n |\hat{\tau}_i - \tau_i| + C_n^e(k) + \sqrt{\frac{k}{n}}.$$

Hence, considering the above inequality and taking expectations in [\(16\)](#) yields

$$E[W_{\mathcal{F}_k}^* - R_n^e(\hat{f}_n^e)] \leq E\left[\frac{1}{n} \sum_{i=1}^n |\hat{\tau}_i - \tau_i|\right] + E[C_n^e(k)] + \sqrt{\frac{k}{n}},$$

and thus by Assumption 3.7

$$E[W_{\mathcal{F}_k}^* - R_n^e(\hat{f}_n^e)] \leq O(\phi_n^{-1}) + E[C_n^e(k)] + \sqrt{\frac{k}{n}}. \quad (17)$$

We now consider the second term on the right hand side of (15). Let \hat{k} be the class k such that

$$\hat{f}_n^e = \hat{f}_{n,\hat{k}}^e.$$

Note that \hat{k} is random. We have

$$R_n^e(\hat{f}_n^e) - W(\hat{f}_n^e) = W_n^e(\hat{f}_{n,\hat{k}}^e) - C_n^e(\hat{k}) - \sqrt{\frac{\hat{k}}{n}} - W(\hat{f}_{n,\hat{k}}^e).$$

By adding and subtracting $W_n(\hat{f}_{n,\hat{k}}^e)$ and the function $\tilde{C}_n(\hat{k})$, we get

$$\begin{aligned} & W_n^e(\hat{f}_{n,\hat{k}}^e) - C_n^e(\hat{k}) - \sqrt{\frac{\hat{k}}{n}} - W(\hat{f}_{n,\hat{k}}^e) = \\ & \left(W_n^e(\hat{f}_{n,\hat{k}}^e) - W_n(\hat{f}_{n,\hat{k}}^e) \right) + \left(\tilde{C}_n(\hat{k}) - C_n^e(\hat{k}) \right) + \left(W_n(\hat{f}_{n,\hat{k}}^e) - W(\hat{f}_{n,\hat{k}}^e) - \tilde{C}_n(\hat{k}) - \sqrt{\frac{\hat{k}}{n}} \right). \end{aligned} \quad (18)$$

Note again that

$$\sup_k \left(W_n^e(\hat{f}_{n,k}^e) - W_n(\hat{f}_{n,k}^e) \right) \leq \frac{1}{n} \sum_{i=1}^n |\hat{\tau}_i - \tau_i|,$$

and so by Assumptions 3.7 and 3.8, the first two terms of (18) are of order $O(\phi_n^{-1})$ in expectation. By the first part of Assumption 3.8, and an argument similar to the one used in the proof of Lemma A.2, it can be shown that

$$E \left[\sup_k \left(W_n(\hat{f}_{n,k}^e) - W(\hat{f}_{n,k}^e) - \tilde{C}_n(k) - \sqrt{\frac{k}{n}} \right) \right] \leq \sqrt{\frac{\log(\Delta e)}{2c_0 n}},$$

where Δ and c_0 are the same constants that appear in A.2. We thus get

$$E[R_n^e(\hat{f}_n^e) - W(\hat{f}_n^e)] \leq O(\phi_n^{-1}) + \sqrt{\frac{\log(\Delta e)}{2m}}. \quad (19)$$

Now combining (17) and (19), we conclude that

$$E[W_{\mathcal{F}_k}^* - W(\hat{f}_n^e)] \leq O(\phi_n^{-1}) + E[C_n^e(k)] + \sqrt{\frac{k}{n}} + \sqrt{\frac{\log(\Delta e)}{2m}}.$$

Finally, by Assumption 3.8, we get

$$E[W_{\mathcal{F}}^* - W(\hat{f}_n^e)] \leq O(\phi_n^{-1}) + E[\tilde{C}_n(k)] + W_{\mathcal{F}}^* - W_{\mathcal{F}_k}^* + \sqrt{\frac{k}{n}} + \sqrt{\frac{\log(\Delta e)}{2m}},$$

for all k , and hence the result follows. ■

Proof of Lemma 3.3 and 3.4. In what follows, we verify that the third condition of Assumption 3.8 is satisfied for the holdout and Rademacher penalties with estimated propensity scores, as the first two conditions follow from previous arguments. If we let

$$\tilde{C}_n(k) = E_\sigma \left[2 \sup_{f \in \mathcal{F}_k} \frac{1}{n} \sum_{i=1}^n \sigma_i f(Z_i) | Z_1, Z_2, \dots, Z_n \right] ,$$

which is the infeasible Rademacher penalty that depends on the unknown propensity score, then it can be shown that

$$|\tilde{C}_n(k) - C_n^e(k)| \leq E_\sigma \left[\frac{2}{n} \sum_{i=1}^n |\hat{\tau}_i - \tau_i| | Z_1, Z_2, \dots, Z_n \right] .$$

Since the right hand side does not depend on k , we conclude that

$$E \sup_{k \geq 1} |\tilde{C}_n(k) - C_n^e(k)| \leq 2E \sum_{i=1}^n |\hat{\tau}_i - \tau_i| = O(\phi_n^{-1}) ,$$

by Assumption 3.7. In the case of the holdout penalty, we can set

$$\tilde{C}_m(k) = W_m(\hat{f}_{m,k}^e) - W_r(\hat{f}_{m,k}^e) .$$

Note that since the propensity score is unknown, the empirical welfare criteria W_m and W_r are infeasible. It can easily be shown that for this choice of $\tilde{C}_m(k)$, we have

$$|\tilde{C}_m(k) - C_m^e(k)| \leq \frac{1}{m} \sum_{i=1}^m |\hat{\tau}_i^E - \tau_i| + \frac{1}{r} \sum_{i=m+1}^n |\hat{\tau}_i^T - \tau_i| ,$$

which yields

$$E \sup_{k \geq 1} |\tilde{C}_m(k) - C_m^e(k)| = O(\phi_n^{-1}) .$$

■

Next, we prove Proposition 5.1.

Let \mathcal{G} be the set of monotone allocations. Let π_k denote the partition of $[0, 1]$ formed by the points $x_i = i/2^k$, $i = 0, \dots, 2^k$. By definition, for each $G \in \mathcal{G}$, there is an associated function $b_G : [0, 1] \rightarrow [0, 1]$ which determines the boundary of the allocation region, that is, such that $G = \{(x_1, x_2) \in \mathcal{X} : x_2 \leq b_G(x_1)\}$. Let $\{\mathcal{G}_k\}_k$ be the approximating sequence defined in Example 3.2, and define $G^* \in \mathcal{G}$ to be a set such that $W(G^*) = W_{\mathcal{G}}^*$ (if no such G^* exists, the argument proceeds by considering an “almost maximizer”).

Proof of Proposition 5.1. Fix some $P \in \mathcal{P}_r$, where \mathcal{P}_r is as defined in Assumption 5.1. By definition,

$$W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^* \leq W(G^*) - W(\tilde{G}_k) ,$$

where $\tilde{G}_k \in \mathcal{G}_k$ is the allocation such that $b_{\tilde{G}_k}(\cdot)$ is the linear interpolation of b_{G^*} on the partition π_k . We can re-write this as

$$\begin{aligned} W(G^*) - W(\tilde{G}_k) &= E \left[\left(\frac{YD}{e(X)} - \frac{Y(1-D)}{1-e(X)} \right) \cdot \left(\mathbf{1}\{X \in G^*\} - \mathbf{1}\{X \in \tilde{G}_k\} \right) \right] \\ &\leq \frac{M}{\kappa} P(G^* \Delta \tilde{G}_k) , \end{aligned} \quad (20)$$

where Δ denotes the symmetric difference operator, $A \Delta B := A \setminus B \cup B \setminus A$. By Assumption 5.1, X has density p_X with respect to Lebesgue measure on $[0, 1]^2$ such that p_x is bounded by some constant A , so that

$$P(G^* \Delta \tilde{G}_k) \leq A \int_0^1 |b_{G^*}(x) - b_{\tilde{G}_k}(x)| dx .$$

We thus conclude that if $b_{\tilde{G}_k}$ is a good L^1 -approximation of b_{G^*} , then the welfare difference $W(G^*) - W(\tilde{G}_k)$ is small. To that end, it remains to bound the approximation bias. Let

$$M_i = [x_{i-1}, x_i] \times [b_{G^*}(x_{i-1}), b_{G^*}(x_i)] ,$$

for $i = 1, \dots, 2^k$. It follows from the monotonicity of b_{G^*} that the graphs of the restrictions of $b_{G^*}(\cdot)$ and $b_{\tilde{G}_k}(\cdot)$ to $[x_{i-1}, x_i]$ are contained in M_i . Hence we have that

$$\int_0^1 |b_{G^*}(x) - b_{\tilde{G}_k}(x)| dx \leq \sum_{i=1}^{2^k} \text{area}(M_i) .$$

Now note that

$$\sum_{i=1}^{2^k} \text{area}(M_i) = \sum_{i=1}^{2^k} |b_{G^*}(x_i) - b_{G^*}(x_{i-1})| \cdot |x_i - x_{i-1}| = \frac{1}{2^k} \sum_{i=1}^{2^k} |b_{G^*}(x_i) - b_{G^*}(x_{i-1})| .$$

By monotonicity, it is the case that

$$\frac{1}{2^k} \sum_{i=1}^{2^k} |b_{G^*}(x_i) - b_{G^*}(x_{i-1})| \leq \frac{1}{2^k} ,$$

since by definition $b_{G^*} : [0, 1] \rightarrow [0, 1]$. We thus obtain that

$$W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^* \leq A \frac{M}{\kappa} 2^{-k} ,$$

as desired. ■

Next, we prove Proposition 5.2. Define

$$N_{\mathcal{G}}(x_1, \dots, x_n) = |\{\{x_1, x_2, \dots, x_n\} \cap G : G \in \mathcal{G}\}| ,$$

then we present the following lemma, which is proved in Györfi et al. (1996):

Lemma A.3. Let \mathcal{G} be the set of monotone allocations. If X has a bounded density with respect to Lebesgue measure on $[0, 1]^2$, then

$$E[N_{\mathcal{G}}(X_1, \dots, X_n)] \leq e^{\alpha\sqrt{n}} .$$

for some constant α .

Proof. See Theorem 13.13 and the discussion following the proof in Györfi et al. (1996). ■

Proof of Proposition 5.2. By Corollary 3.4 in Geer (2000), we have that

$$P\left(\sup_{f \in \mathcal{F}} |W_n(f) - W(f)| > \epsilon\right) \leq 4P\left(\sup_{f \in \mathcal{F}} \left|\frac{1}{n} \sum_{i=1}^n \sigma_i f(Z_i)\right| > \frac{\epsilon}{4}\right) ,$$

for $\epsilon \geq \sqrt{\frac{8(M/\kappa)^2}{n}}$, where σ_i are Rademacher random variables (this follows from two symmetrizations). Write $f(Z)$ as

$$f_G(Z) = g(Z)\mathbf{1}\{X \in G\} ,$$

where $g(Z) = \left(\frac{YD}{e(X)} - \frac{Y(1-D)}{1-e(X)}\right)$. Conditioning on $\{Z_i = z_i = (y_i, x_i, d_i)\}_{i=1}^n$, and applying the union bound, we get that

$$\begin{aligned} & P\left(\sup_{G \in \mathcal{G}} \left|\frac{1}{n} \sum_{i=1}^n \sigma_i g(Z_i)\mathbf{1}\{X_i \in G\}\right| > \frac{\epsilon}{4} \middle| \{Z_i = z_i\}_{i=1}^n\right) \leq \\ & N_{\mathcal{G}}(x_1, \dots, x_n) \sup_{G \in \mathcal{G}} P\left(\left|\frac{1}{n} \sum_{i=1}^n \sigma_i g(Z_i)\mathbf{1}\{X_i \in G\}\right| > \frac{\epsilon}{4} \middle| \{Z_i = z_i\}_{i=1}^n\right) . \end{aligned} \quad (21)$$

By Hoeffding's inequality,

$$P\left(\left|\frac{1}{n} \sum_{i=1}^n \sigma_i g(Z_i)\mathbf{1}\{X_i \in G\}\right| > \frac{\epsilon}{4} \middle| \{Z_i = z_i\}_{i=1}^n\right) \leq 2e^{-n\epsilon^2/c} ,$$

where $c = (4M/\kappa)^2$. Taking expectations, we can conclude that

$$P\left(\sup_{f \in \mathcal{F}} \left|\frac{1}{n} \sum_{i=1}^n \sigma_i f(Z_i)\right| > \frac{\epsilon}{4}\right) \leq 2E[N_{\mathcal{G}}(X_1, \dots, X_n)]e^{-n\epsilon^2/c} .$$

Using Lemma A.3, we get that

$$P\left(\sup_{f \in \mathcal{F}} |W_n(f) - W(f)| > \epsilon\right) \leq 8e^{\alpha\sqrt{n}}e^{-n\epsilon^2/c} ,$$

for $\epsilon \geq \sqrt{\frac{8(M/\kappa)^2}{n}}$. Let $\epsilon^*(n) = \sqrt{\frac{8(M/\kappa)^2}{n}}$, then the result follows by a slight modification of the integration argument presented in Problem 12.1 of Györfi et al. (1996) (split the integral of the tail probability as follows: $\int_0^\infty = \int_0^{\epsilon^*(n)} + \int_{\epsilon^*(n)}^u + \int_u^\infty$, bound the first integral by $\epsilon^*(n)$, the second by u , and the third by our tail inequality, and proceed analogously). ■

B Supplementary Results

B.1 Supplement to Example 2.3

We work through the claim of Example 2.3 in detail. Suppose the outcomes of interest to the planner are described by

$$Y(k) = g(k, A) - \mathbf{1}\{k = 1\}c ,$$

where A is an unobserved measure of a student's ability, and c is the per-unit cost of the scholarship to the planner. Let

$$h(a) = g(1, a) - g(0, a) .$$

Suppose the planner has two covariates $X = (Z, T)$, on which to base treatment, where Z is parental income and T is a student's GPA. Define

$$\tau(t, z) := E[h(A)|Z = z, T = t] = \int h(a)dF_{A|Z,T}(a|z, t) ,$$

to be the average treatment effect (ignoring costs) conditional on $Z = z, T = t$ (note that if we consider Assumption 3.1 then $h(A)$ has finite support, which guarantees the existence of τ). The unrestricted optimal allocation is given by

$$G_{FB}^* := \{(z, t) : \tau(z, t) \geq c\} .$$

We claimed in Example 2.3 that some plausible assumptions about $h(\cdot)$ and (A, T, Z) could give rise to an optimal allocation which is *monotone*, as defined in Example 2.3.

First, the planner makes the following assumption on $h(\cdot)$:

Assumption B.1. $h(a)$ is increasing in a .

This assumption asserts that the function g has *increasing differences*, which is a common assumption made in economics when doing comparative statics analysis. Intuitively, it says that higher ability students will realize a larger difference in outcomes if they receive the scholarship than lower ability students.

Next, the planner makes the following assumptions about the conditional distribution of $(A|Z, T)$:

Assumption B.2. (FOSD of A in (Z, T))

- $F_{A|Z,T}(\cdot|z, t) \succeq^{FOSD} F_{A|Z,T}(\cdot|z, t')$ for $t \geq t'$
- $F_{A|Z,T}(\cdot|z, t) \succeq^{FOSD} F_{A|Z,T}(\cdot|z', t)$ for $z \leq z'$

Stochastic-dominance assumptions of this type have been employed by, for example, [Blundell et al. \(2007\)](#) in the study of wage distributions. Intuitively, Assumption [B.2](#) asserts that, given a fixed level of parental income, a higher GPA is an indication of higher innate ability, and that given a fixed GPA, lower levels of parental income are an indication of higher innate ability. An assumption of this type could come out of a production function for cognitive achievement, for example as studied in [Todd and Wolpin \(2003\)](#).

Given these assumptions, we can show that $\tau(z, t) \geq \tau(z, t')$ if $t \geq t'$, and $\tau(z, t) \geq \tau(z', t)$ if $z \leq z'$. This follows by the fact that, for an increasing function $f(\cdot)$ and two distributions G_1 and G_2 , such that G_1 first order stochastically dominates G_2 , it is the case that

$$\int f dG_1 \geq \int f dG_2 .$$

This establishes that the first best allocation is indeed monotone.

B.2 Supplement to Example [2.1](#)

We elaborate on the example introduced in Example [2.1](#). We construct an approximating sequence that results in what [Scott and Nowak \(2002\)](#) call a *dyadic decision tree*. From now on assume it is the case that $\mathcal{X} = [0, 1]^d$. First, we define a *sequential dyadic partition* (SDP). Let $\{R_1, R_2, \dots, R_k\}$ be a partition of the the covariate space where each R_i is a hyper-rectangle with sides parallel to the co-ordinate axes. Given a cell R_i , let $R_i^{(1,j)}$ and $R_i^{(2,j)}$ be the hyper-rectangles formed by splitting R_i at its midpoint along the co-ordinate j . A SDP is defined recursively as follows:

- The trivial partition $\{[0, 1]^d\}$ is a SDP
- If $\{R_1, R_2, \dots, R_k\}$ is a SDP, then so is

$$\{R_1, \dots, R_{i-1}, R_i^{(1,j)}, R_i^{(2,j)}, R_{i+1}, \dots, R_k\} ,$$

where $1 \leq i \leq k$ and $1 \leq j \leq d$.

In words, a SDP is formed by recusively splitting a hyper-cube at its midpoint on some coordinate. A *dyadic decision tree* (DDT) with k splits is a SDP with k partitions, paired with a $\{0, 1\}$ label for each hyper-rectangle in the SDP. Given a DDT T_k with k splits, let $G(T_k)$ be the set of covariate points in \mathcal{X} such that those covariates are labeled with a 1 in T_k . Our approximating class is defined as follows:

$$\mathcal{G}_k = \{G \subset \mathcal{X} : G = G(T_k) \text{ for some DDT } T_k \text{ with } k \text{ splits}\} .$$

It follows by results in [Scott and Nowak \(2002\)](#) that \mathcal{G}_k has finite VC dimension. Given this approximating sequence, the PWM procedure can be applied to choose the appropriate DDT.

Kallus (2016) develops Optimal Personalization Trees, which solve a similar problem to the EWM problem for a given class \mathcal{G}_k . We expect that a modification of his optimization problem would be appropriate in our setting.

We expect that under appropriate regularity conditions we could derive bounds on the maximum regret of this version of PWM with respect to the *unrestricted* optimum. The first question one might ask is how the bounds on maximum regret of PWM with this approximating sequence would compare to the bounds on maximum regret that exist for plug-in rules. As discussed in Kitagawa and Tetenov (2015), if the plug-in rule is implemented with appropriate local-polynomial estimators, and smoothness conditions on the *regression functions* $E(Y(d)|X = x)$ are imposed, a bound on maximum regret can be derived. On the other hand, as explained in Audibert et al. (2007) in the context of classification, although results for plug-in rules typically require assumptions on smoothness of the regression functions, the analogues to our approach in classification typically require regularity conditions on the *boundary* of the decision set G_{FB}^* . In this sense, a comparison of the regularity conditions for plug-in rules and PWM-type rules would suggest that they are complementary approaches.

Where our method could provide a benefit over a plug-in rule is in the ease of interpretation of the resulting treatment allocation. Kitagawa and Tetenov (2015) show in their application that plug-in rules that use popular non-parametric estimators of τ can result in treatment allocations that, from a practical perspective, may be hard to implement and interpret. On the other hand, the allocations that result from our rule partition the covariate space in such a way that they can be interpreted as a series of “yes-or-no” questions, which is a primary reason for the popularity of decision trees in classification.

B.3 Supplement to Remark 3.2

The demeaned EWM rule is defined as follows: Let $Y_i^{dm} := Y_i - E_n[Y_i]$, then the demeaned EWM rule solves the following problem:

$$\max_{G \in \mathcal{G}} E_n \left[\frac{Y_i^{dm} D_i}{e(X_i)} \mathbf{1}\{X_i \in G\} + \frac{Y_i^{dm} (1 - D_i)}{1 - e(X_i)} \mathbf{1}\{X_i \in G\} \right] .$$

In this section we redo all of the computations of Sections 4 and 5, where we use this demeaned version of EWM in each step of the Penalized Welfare Maximization process. In Figure 6 we reproduce the exercise of Section 4. Note that moving to the demeaned version of EWM does not make any qualitative changes to the estimated regret.

[Figure 6 about here.]

In Figures 7, 8, 9, we reproduce the exercise of Section 5. What is interesting to note is that PWM selects the fourth class in the sequence when using the demeaned version, whereas PWM selected the second class in the sequence for the original problem.

[Figure 7 about here.]

[Figure 8 about here.]

[Figure 9 about here.]

B.4 Supplement to Remarks 3.5 and 3.6

In this subsection we provide some simple calculations that justify the comments made in Remarks 3.5 and 3.6. Consider first the Rademacher penalty, then Proposition 3.2 shows that

$$E_{P^n}[W_{\mathcal{G}}^* - W(\hat{G}_n)] \leq \inf_k \left[C \frac{M}{\kappa} \sqrt{\frac{V_k}{n}} + (W_{\mathcal{G}}^* - W_{\mathcal{G}_k}^*) + \sqrt{\frac{k}{n}} \right] + g(M, \kappa) \frac{M}{\kappa} \sqrt{\frac{1}{n}},$$

where C is the universal constant derived in the bound of EWM in Kitagawa and Tetenov (2015) and g is defined as

$$g(M, \kappa) := 6 \sqrt{\log \left(\frac{3\sqrt{e} M}{\sqrt{2} \kappa} \right)}.$$

Our first task is to quantify the size of C . By the proof of Lemma A.4. in Kitagawa and Tetenov (2015), we can see that the constant C depends on a universal constant K derived in Theorem 2.6.7 of Van Der Vaart and Wellner (1996), which establishes a bound on the covering numbers of a VC subgraph class. Inspection of the proof in Van Der Vaart and Wellner (1996) allows us to conclude that a suitable K is given by $K = 3\sqrt{e}/8$. Plugging this in to the expression for C derived in Kitagawa and Tetenov (2015) allows us to conclude that a suitable C is given by $C = 36.17$. Turning to $g(M, \kappa)$, we can calculate that in order for it to surpass C by an order of magnitude, we would need M/κ to be about as large as 10^{120} . This give us a sense of the relative sizes of the terms in our bound.

C Computational Details

In this section we provide details on how we perform the computations of Sections 4 and 5. All of our work is implemented in Python 2.7 paired with Gurobi. To clarity the exposition, we begin with Section 5, which is more straightforward, then proceed to Section 4.

C.1 Application Details

We will now describe how we compute each $\hat{G}_{n,k}$ to solve PWM over monotone allocations. Recall the definition of $\psi_{T,j}(x)$ as defined in Example 3.2. We modify this definition to accommodate the fact that our covariates do not lie in the unit interval. In particular, we restrict ourselves to levels of education that lie in the interval $[5, 20]$, which leads to the following modification.

$$\psi_{T,j}(x) = \begin{cases} 1 - |\frac{T}{15}(x - 5) - j|, & x \in [\frac{j-1}{T/15} + 5, \frac{j+1}{T/15} + 5] \cap [5, 20] \\ 0, & \text{otherwise} . \end{cases}$$

Let $\Theta_T = [\theta_0 \ \theta_1 \ \dots \ \theta_T]'$ and let $\bar{\Theta}_T = [-1 \ \theta_0 \ \theta_1 \ \dots \ \theta_T]'$. Let our two dimensional covariate be denoted as $x = (x^{(1)}, x^{(2)})$ where $x^{(1)}$ is level of education and $x^{(2)}$ is previous earnings. Let

$$\Psi_T(x) = [x^{(2)} \ \psi_{T,0}(x^{(1)}) \ \dots \ \psi_{T,T}(x^{(1)})]'$$

To compute $\hat{G}_{n,k}$ we solve the following mixed integer linear program (MILP), which modifies the MILP described in Kitagawa and Tetenov (2015) for “Single Linear Index Rules”:

$$\begin{aligned} & \max_{\substack{\theta_0, \theta_1, \dots, \theta_T, \\ z_1, \dots, z_n}} && \sum_{i=1}^n \tau_i \cdot z_i \\ \text{subject to} && \frac{\bar{\Theta}_T' \Psi_T(x_i)}{c_T} < z_i \leq \frac{\bar{\Theta}_T' \Psi_T(x_i)}{c_T} + 1, \ i = 1, \dots, n \\ && z_i \in \{0, 1\}, \ i = 1, \dots, n \\ && D_T \Theta_T \geq 0 \end{aligned}$$

where $T = 2^{k-1}$, τ_i is as defined in equation (2), c_T is an appropriate constant (to be discussed in the following sentence), and D_T is the differentiation matrix as defined in Example 3.2. c_T is a constant chosen such that $c_T > \sup_{\Theta_T} |\bar{\Theta}_T' \Psi_T(x_i)|$, which allows us to formulate a set of what are known as “big-M” constraints. To implement such a constraint it must necessarily be the case that Θ_T is bounded, so in order to implement PWM we also include an implicit (very large) bound on the possible treatment allocations.¹⁰

The first two sets of constraints impose that the treatment allocation result in a piecewise linear boundary, the third set of constraints impose that this boundary is monotone. The strength of this formulation is that it imposes monotonicity via a *linear* constraint, which allows us to solve the problem as a MILP.

¹⁰Big-M constraints have the potential to cause numerical instabilities when solving MILPs that are poorly formulated. We found that it was important to ensure that the covariates are scaled to within the same order of magnitude and that the `IntFeasTol` and `FeasibilityTol` parameters in Gurobi were set to their smallest possible values.

C.2 Simulation Details

We describe how we compute each $\hat{G}_{n,k}$ to solve PWM over threshold allocations on d covariates. Define x to be a $(d+1)$ -dimensional vector where $x = (1, x^{(1)}, x^{(2)}, \dots, x^{(d)})$, with the last d components denoting the d covariates, and suppose $x \in [0, 1]^{d+1}$, which is the case in the simulation design. We define the threshold β_k on covariate $x^{(k)}$ to be a $(d+1)$ -dimensional vector such that the first component is in $[-1, 1]$, all other components other than the $(k+1)$ st are zero, and the $(k+1)$ st component is one of $\{1, -1\}$. To compute $\hat{G}_{n,k}$ we compute a series of mixed-integer linear programs (MILPs). Let $A \subseteq \{1, 2, \dots, d\}$ denote the subset of “active” covariates, then the optimization problem that solves for threshold allocations over the set of covariates A modifies the MILP described in [Kitagawa and Tetenov \(2015\)](#) for “Multiple Linear Index Rules”:

$$\begin{aligned}
& \max_{\substack{\{\beta_a\}_{a \in A}, \\ \{z_1^a, \dots, z_n^a\}_{a \in A}, z_1^*, \dots, z_n^*}} && \sum_{i=1}^n \tau_i \cdot z_i^* \\
\text{subject to} && \frac{x'_i \beta_a}{c} < z_i^a \leq \frac{x'_i \beta_a}{c} + 1, \quad i = 1, \dots, n, \quad a \in A \\
&& 1 - |A| + \sum_{a \in A} z_i^a \leq z_i^* \leq \frac{1}{|A|} \sum_{a \in A} z_i^a, \quad i = 1, \dots, n \\
&& \{z_i^a\}_{a \in A}, z_i^* \in \{0, 1\}, \quad i = 1, \dots, n \\
&& \beta_a^{(1)} \in [-1, 1], \quad a \in A \\
&& \beta_a^{(a+1)} \in \{-1, 1\}, \quad a \in A \\
&& \beta_a^{(k)} = 0, \quad k > 1, k \neq a+1, \quad a \in A
\end{aligned}$$

The last three constraints impose that β_a satisfies the conditions of a threshold as stated above, and can be formulated into a MILP by adding auxiliary variables. Again we require an appropriately chosen constant c to implement a set of big-M constraints, but in this case the choice is straightforward: $c = d + 2$ will suffice since this guarantees that $c > x'_i \beta_a$ for any possible x_i and β_a , by construction. $\hat{G}_{n,k}$ is computed by solving this problem over all subsets A such that $|A| = k - 1$, and choosing the allocation that results in the highest value of the objective function. This brute-force method involves solving $\binom{d}{k-1}$ MILPs, which could be computationally prohibitive for large d . In our testing, computation of $\hat{G}_{n,k}$ for eight covariates, with $n = 500$ and $k = 5$ took 15 minutes on a 2012 Macbook Pro.

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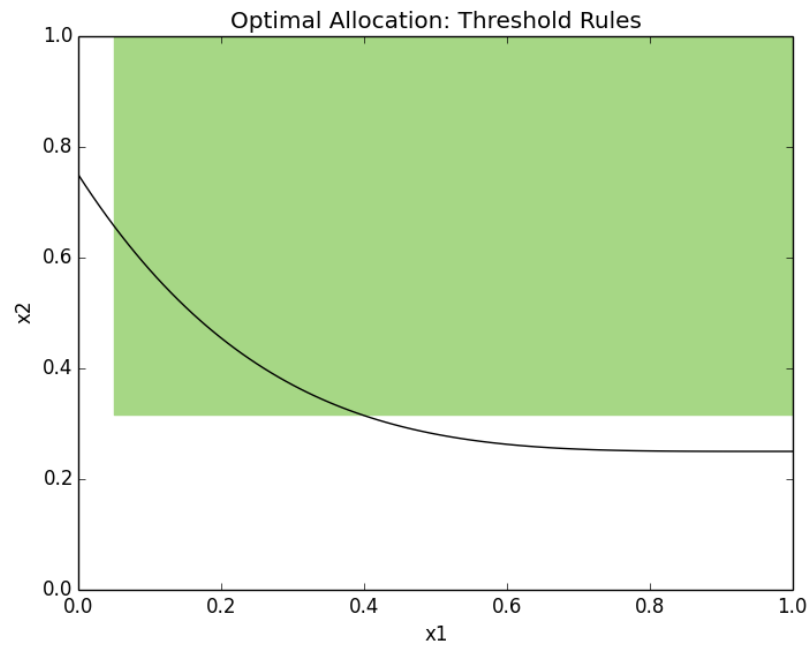


Figure 1: Shaded in green: the best threshold-allocation for our design. Second-best welfare: 29.3
Traced in black: the boundary of the first-best allocation.

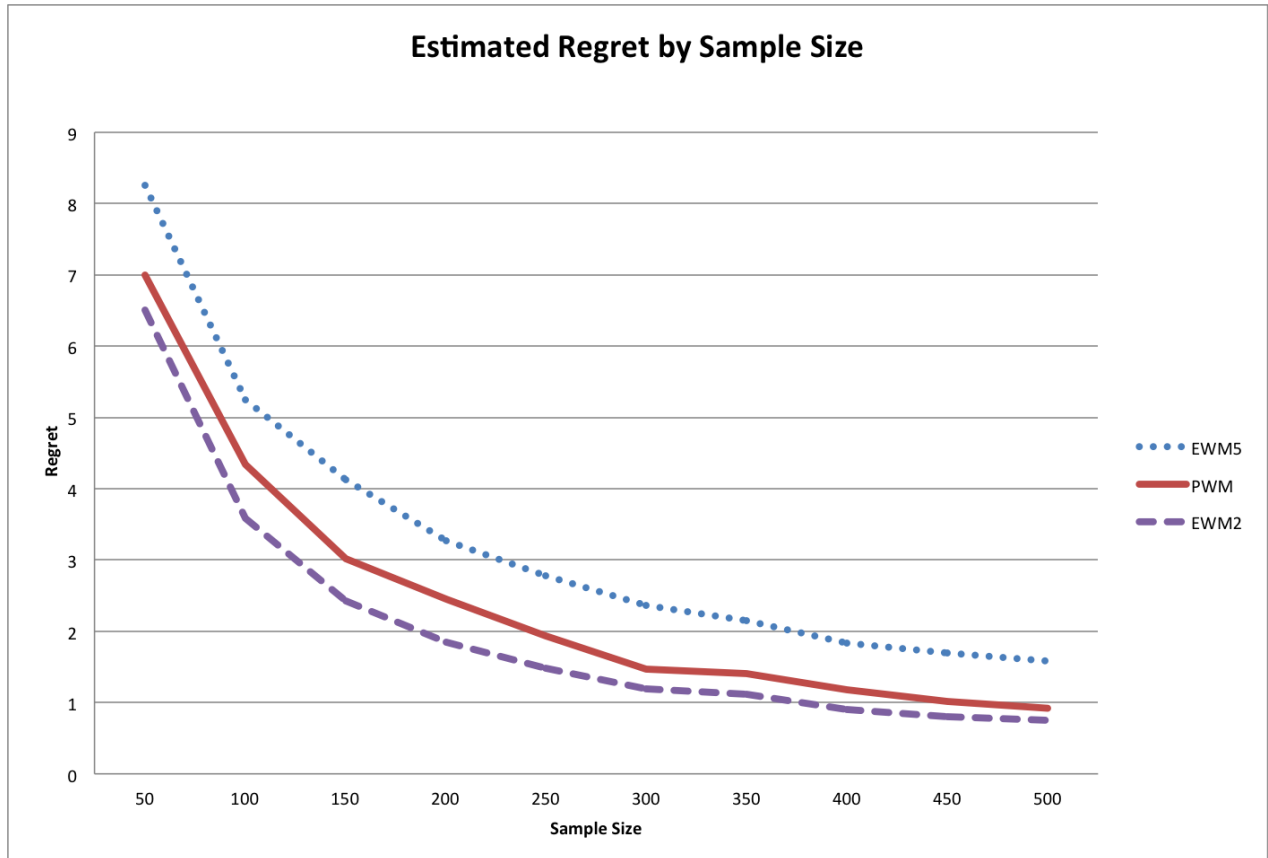


Figure 2: Estimated regret by sample size. Optimal welfare: 29.3. EWM5 corresponds to \mathcal{G}_6 (five covariates), EWM2 corresponds to \mathcal{G}_3 (two covariates).

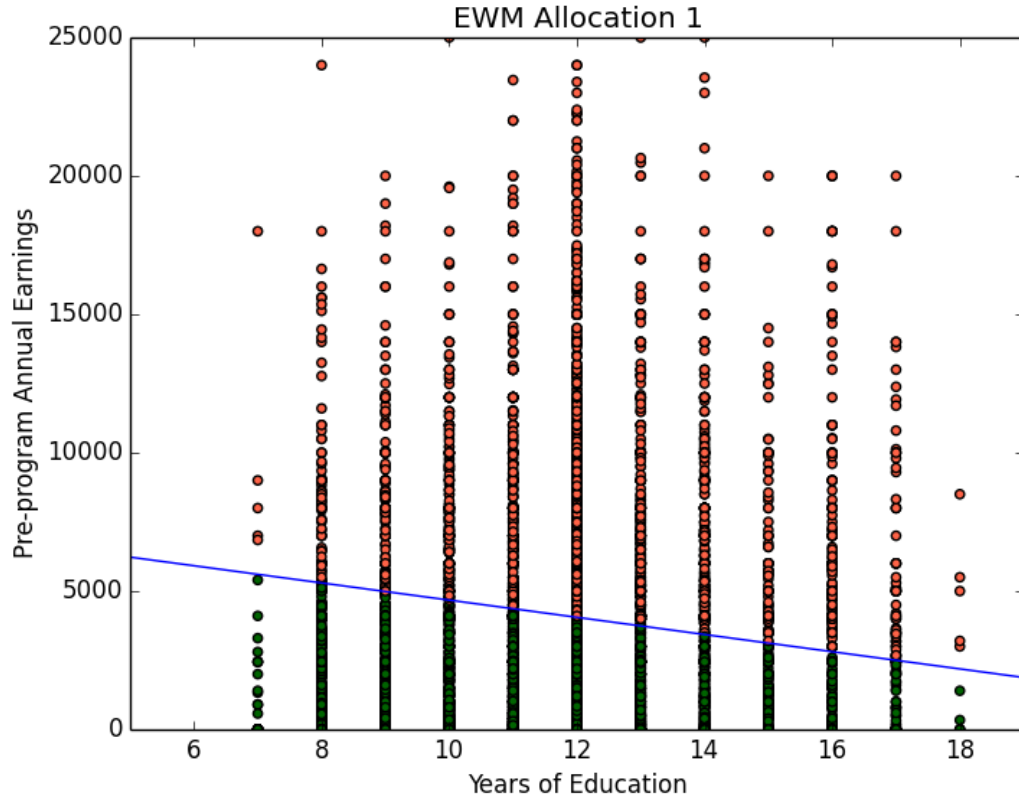


Figure 3: The resulting treatment allocation from performing EWM in \mathcal{G}_1 . Each point represents a covariate pair in the sample. The region shaded in green (dark) is the prescribed treatment region, the region shaded in red (light) is the prescribed control region.

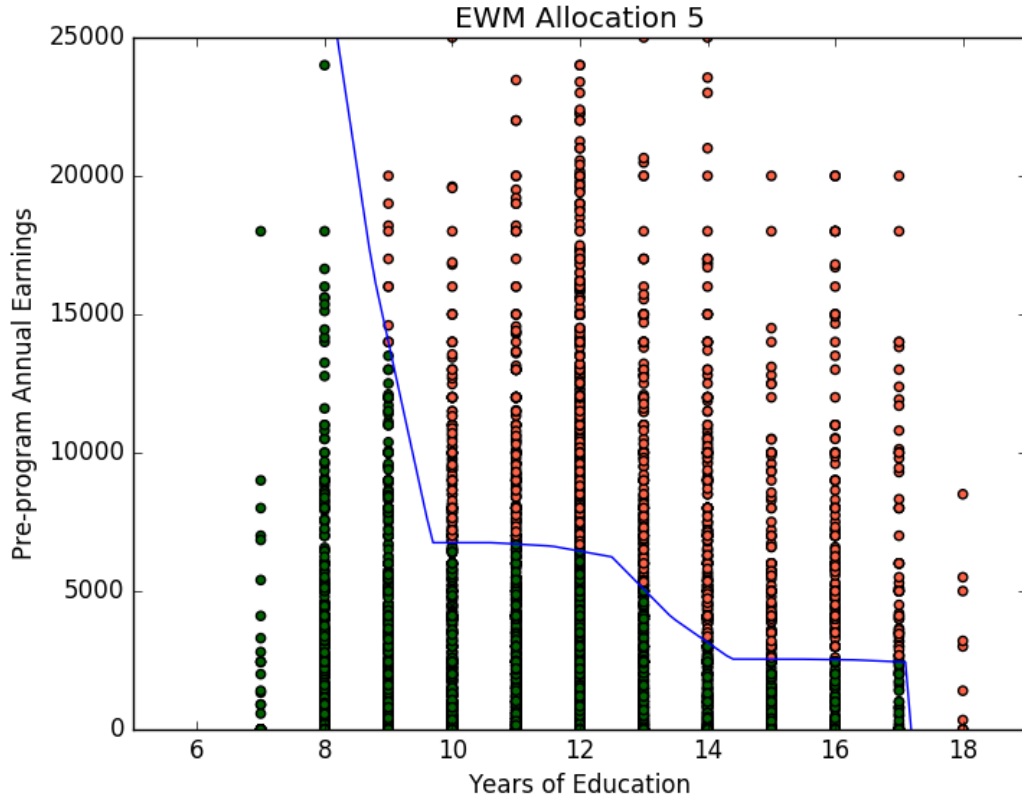


Figure 4: The resulting treatment allocation from performing EWM in \mathcal{G}_5 . Each point represents a covariate pair in the sample. The region shaded in green (dark) is the prescribed treatment region, the region shaded in red (light) is the prescribed control region.

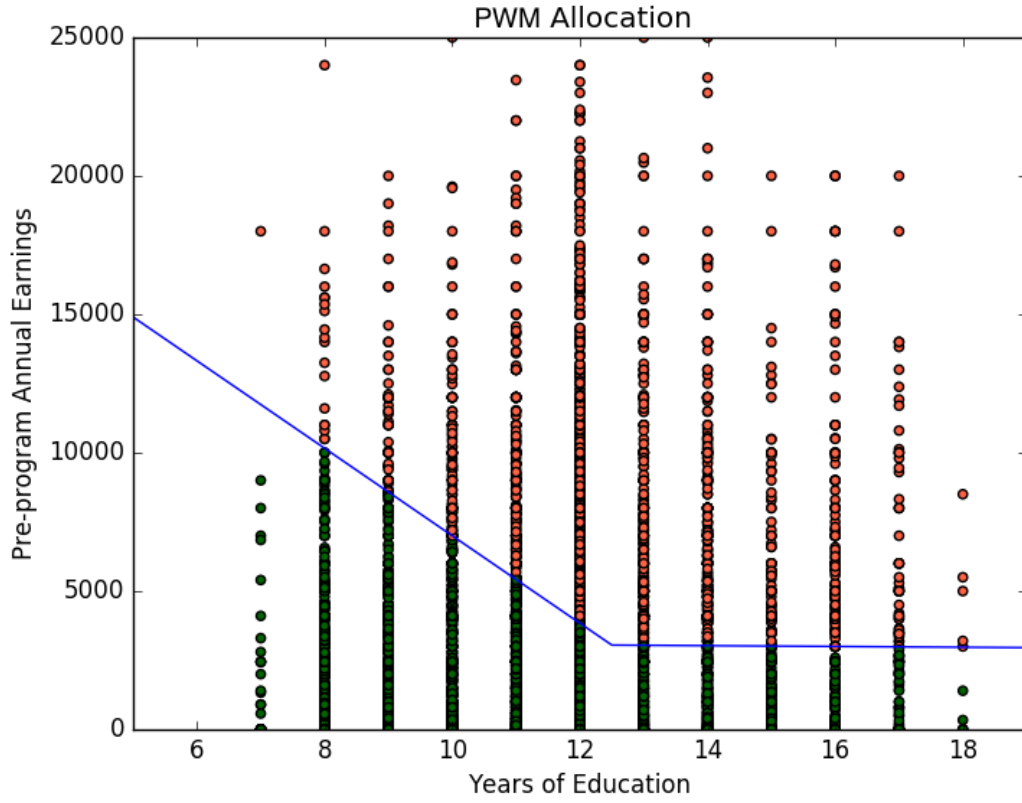


Figure 5: The resulting treatment allocation from performing PWM on the approximating sequence $\{\mathcal{G}_k\}_{k=1}^5$. Each point represents a covariate pair in the sample. The region shaded in green (dark) is the prescribed treatment region, the region shaded in red (light) is the prescribed control region.

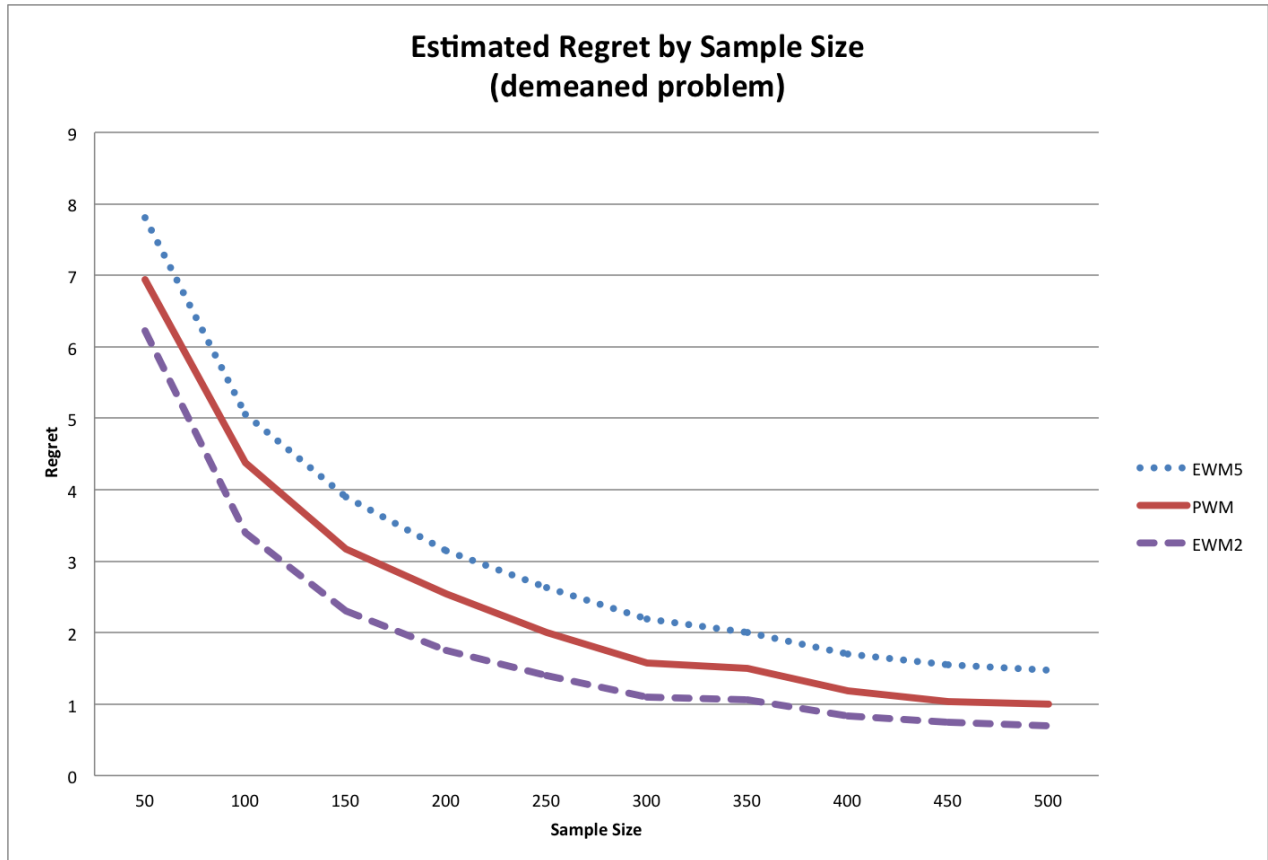


Figure 6: Estimated regret by sample size for the demeaned problem. EWM5 corresponds to \mathcal{G}_6 (five covariates), EWM2 corresponds to \mathcal{G}_3 (two covariates).

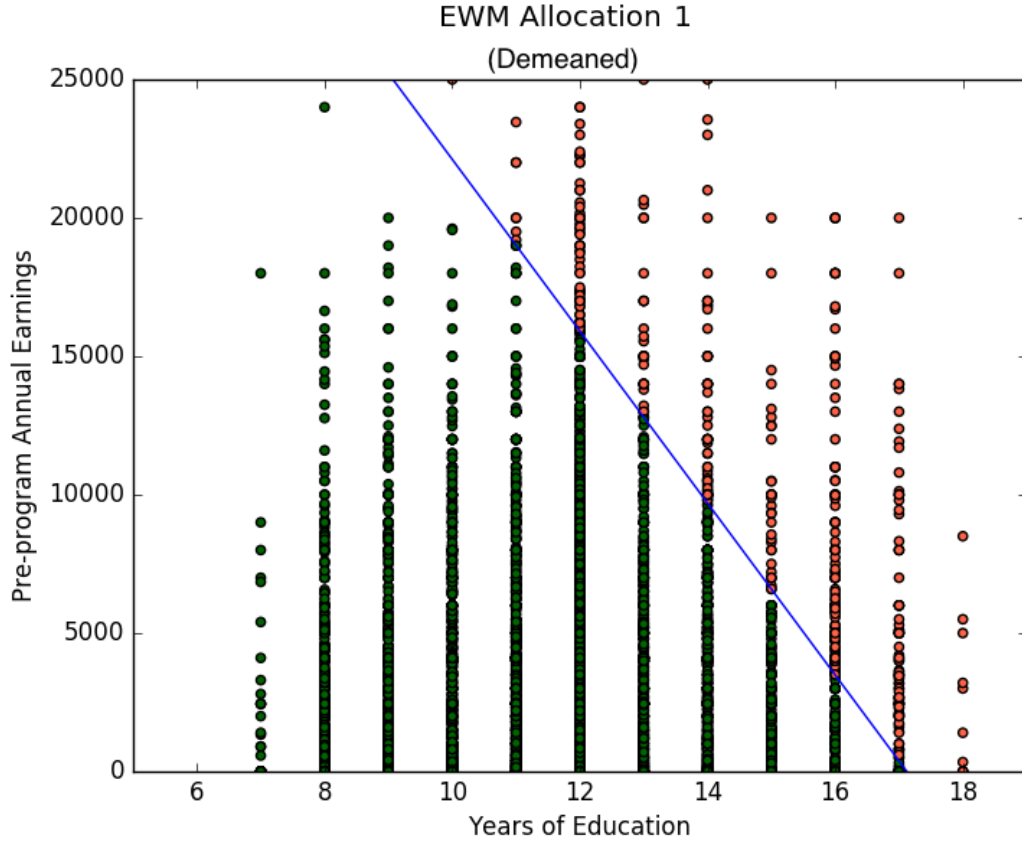


Figure 7: The resulting treatment allocation from performing demeaned EWM in \mathcal{G}_1 . Each point represents a covariate pair in the sample. The region shaded in green (dark) is the prescribed treatment region, the region shaded in red (light) is the prescribed control region.

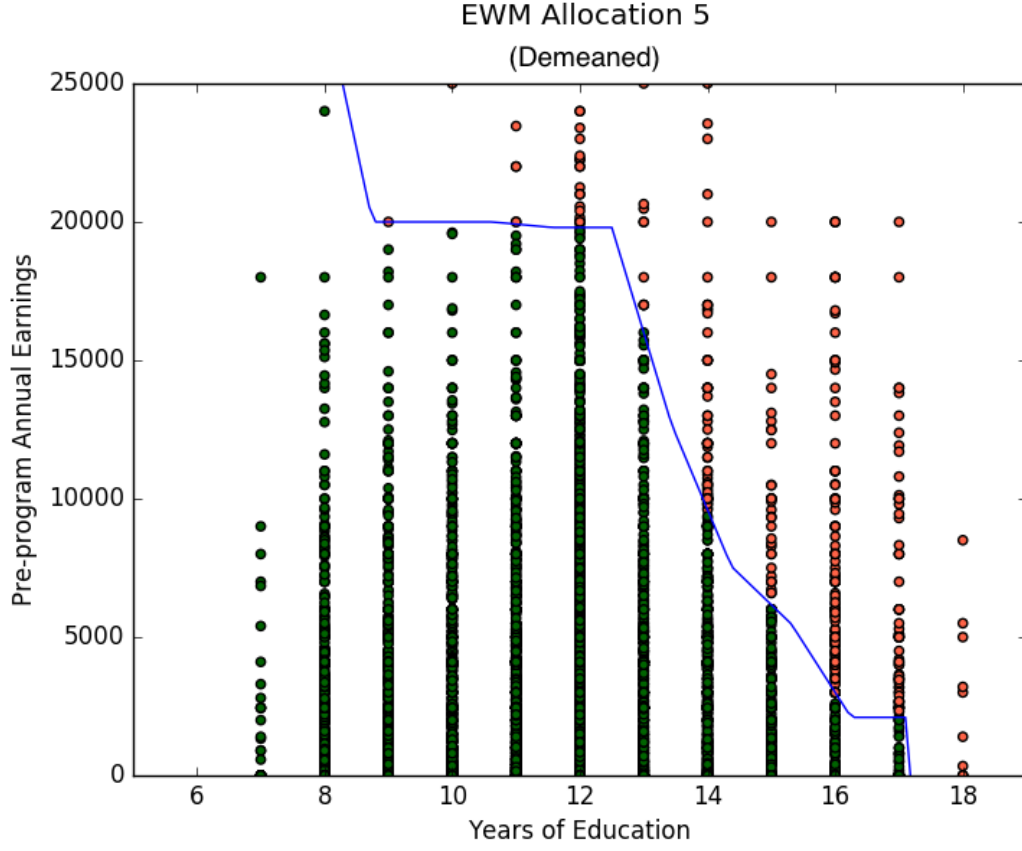


Figure 8: The resulting treatment allocation from performing demeaned EWM in \mathcal{G}_5 . Each point represents a covariate pair in the sample. The region shaded in green (dark) is the prescribed treatment region, the region shaded in red (light) is the prescribed control region.

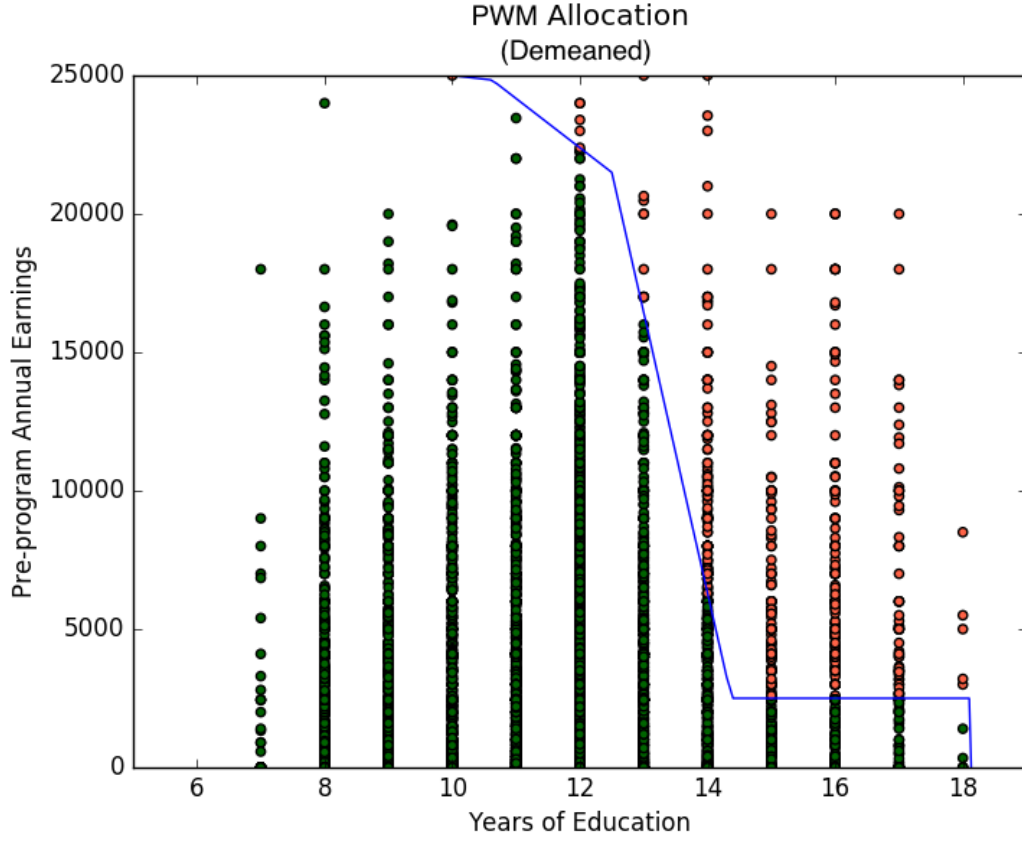


Figure 9: The resulting treatment allocation from performing demeaned PWM on the approximating sequence $\{\mathcal{G}_k\}_{k=1}^5$. Each point represents a covariate pair in the sample. The region shaded in green (dark) is the prescribed treatment region, the region shaded in red (light) is the prescribed control region.